CONSTRUCTION OF GIBBS MEASURES FOR PERIODIC NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. Dispersive PDEs such as nonlinear Schrödinger equations (NLS) have been studied at length using the tools of classical and harmonic analysis. However, periodic NLS for low regularity have been more difficult to investigate using deterministic methods. Instead, probabilistic tools can be used to obtain various well-posedness results. In this expository paper based on the lecture notes of Oh '17 and building toward the results of Bourgain '94, we treat the construction of Gibbs measures on Sobolev spaces $H^\sigma(\mathbb{T})$, which enable the use of probabilistic tools for studying periodic NLS in the low regularity regime.

1. Introduction and Notation

1.1. **Introduction.** The main object of our study will be the nonlinear Schrödinger equation on the d-dimensional torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$,

$$\begin{cases} i\partial_t u + \Delta u = \lambda |u|^{p-1} u \\ u(t_0, x) = u_0(x). \end{cases}$$

Here, the desired solution is the scalar field $u : I \times \mathbb{T}^d \to \mathbb{C}$ for some time interval $I \subset \mathbb{R}$. The initial data u_0 lies in a Sobolev space $H^s_x(\mathbb{T}^d)$, the exponent $1 < p < \infty$ denotes the power of the nonlinearity, and the sign $\lambda \in \{-1, 0, 1\}$ denotes the nature of the nonlinearity (focusing, absent, or defocusing, respectively).

There is extensive theory surrounding local and global well-posedness for aperiodic NLS, i.e. those with domain $\mathbb{R}^d$ rather than $\mathbb{T}^d$. For example, [Tao06, Proposition 3.8] gives us that the aperiodic NLS is locally well-posed in $H^s_x(\mathbb{R}^d)$ for $p > 1$ an odd integer and s sufficiently large. Even in lower regularity, [Tao06, Proposition 3.15] gives local well-posedness in $L^2_x(\mathbb{R}^d)$ for certain choices of p. Conservation laws can allow such local well-posedness results to be extended globally.

Some of these techniques can be applied to periodic NLS for high regularity and weak nonlinearity. See, e.g., [Bou93, Section 4]. Unfortunately, it is not always possible to apply these deterministic techniques to study low regularity well-posedness of NLS in the periodic setting. Instead, we turn to probabilistic methods with the goal of proving almost sure well-posedness with respect to the initial data. This will allow us to disregard a small set of initial data.

Of course, proving almost sure well-posedness requires defining a probability measure on a space of functions on $\mathbb{T}^d$ within which the initial data can lie. To that end, we begin with the goal of defining a measure on $H^\sigma_x(\mathbb{T}^d)$ for certain values of σ. The challenge of defining such a measure is that $H^\sigma_x(\mathbb{T}^d)$ is infinite-dimensional. Unlike the finite-dimensional case sometimes associated with certain ordinary differential equations, we can no longer use the Lebesgue measure because it has no infinite-dimensional version.

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As it happens, the most promising candidates for such measures are Gibbs measures that are formally invariant under the flow of NLS. Roughly speaking, invariance means that the measure of a set of solutions at a certain time is the same as the measure of the initial data that generated those solutions. This is a helpful property because the measure can be interpreted as describing the long-term behavior of solutions and because we can view the system as a dynamical system in which the passage of time is a measure-preserving transformation. This gives us access to powerful recurrence theorems such as those of Poincaré and Furstenberg. See [OQ13, p. 2] for applications.

We will begin with a finite-dimensional example in which we construct invariant Gibbs measures for a Hamiltonian system of ODEs. Then, we formally describe an outline of the construction of Gibbs measures for periodic NLS in one dimension. Finally, we proceed with a rigorous construction of Gibbs measures and prove that they are nontrivial probability measures on $H_x^\sigma(\mathbb{T})$ for sufficiently small σ, at least in the defocusing case. The structure of this paper and most proofs are based on those of [Oh17].

1.2. **Notation.** For $f : \mathbb{T}^d \to \mathbb{C}$, we define $\hat{f} : \mathbb{Z}^d \to \mathbb{C}$ by

$$\hat{f}(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-in\cdot x}\, dx.$$

For any $n \in \mathbb{Z}^d$, we write

$$|n| := \|n\|_2 = \left(\sum_{j=1}^{d} n_j^2 \right)^{\frac{1}{2}}$$

and

$$\langle n \rangle = \sqrt{1 + |n|^2}.$$

We will write $A \lesssim B$ to denote $A \leq CB$ for some constant $C \in (0, \infty)$, $B \gtrsim A$ to denote $A \lesssim B$, and $A \asymp B$ to denote $A \lesssim B \lesssim A$.

2. Gibbs measures in finite dimension

Let us take a brief detour from our discussion of NLS in order to consider a finite-dimensional system. Let $\mathcal{H} : \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian with sufficient regularity and suppose that $\mathcal{H}(p, q) \gtrsim (|p| + |q|)^\delta$ for some $\delta > 0$.[1] Then, Hamilton's equations are

$$\begin{cases} \partial_t p_j = \dfrac{\partial \mathcal{H}}{\partial q_j} \\ \partial_t q_j = -\dfrac{\partial \mathcal{H}}{\partial p_j} \end{cases} \tag{2.1}$$

for $j = 1, \dots, n$.

Consider the vector field $\nabla_\omega \mathcal{H}$ given by

$$\nabla_\omega \mathcal{H} = \left(\frac{\partial \mathcal{H}}{\partial q_1}, \dots, \frac{\partial \mathcal{H}}{\partial q_n}, -\frac{\partial \mathcal{H}}{\partial p_1}, \dots, -\frac{\partial \mathcal{H}}{\partial p_n} \right),$$

which represents the flow generated by (2.1). A simple computation yields that

$$\nabla \cdot \nabla_\omega \mathcal{H} = \sum_{j=1}^{n} \left[\frac{\partial}{\partial p_j} \frac{\partial \mathcal{H}}{\partial q_j} + \frac{\partial}{\partial q_j} \left(-\frac{\partial \mathcal{H}}{\partial p_j} \right) \right] = 0.$$

[1] This coercivity condition ensures that (2.1) is globally well-posed and $e^{-\beta \mathcal{H}}$ is integrable.

Then, Liouville's theorem gives us that the Lebesgue measure

$$dp\,dq = \prod_{j=1}^{n} dp_j\,dq_j$$

is invariant under $\nabla_\omega \mathcal{H}$, the flow generated by (2.1).

We can also compute

$$\begin{aligned} \frac{d}{dt}\mathcal{H}(p(t), q(t)) &= \frac{\partial \mathcal{H}}{\partial p}\,\partial_t p + \frac{\partial \mathcal{H}}{\partial q}\,\partial_t q \\ &= \frac{\partial \mathcal{H}}{\partial p}\,\frac{\partial \mathcal{H}}{\partial q} + \frac{\partial \mathcal{H}}{\partial q}\left(-\frac{\partial \mathcal{H}}{\partial p}\right) \\ &= 0, \end{aligned}$$

so the Hamiltonian is conserved over time.

Together, invariance of the Lebesgue measure and conservation of the Hamiltonian allow us to construct another family of invariant measures, which we call the Gibbs measures.

Definition 2.1. *Given a Hamiltonian system of the form (2.1) and a constant $\beta > 0$, we define the* Gibbs measure μ *by*

$$d\mu = \frac{1}{Z_\beta} e^{-\beta \mathcal{H}(p,q)}\,dp\,dq,$$

where the normalizing constant Z_β (called the "partition function") is given by

$$Z_\beta = \int_{\mathbb{R}^{2n}} e^{-\beta \mathcal{H}(p,q)}\,dp\,dq.$$

It is easy to verify that μ is indeed a probability measure. We will show that Gibbs measures are also invariant under the flow generated by (2.1), a property we make precise with the following definition. In the sequel, we use $\beta = 1$ and $Z := Z_1$, but any choice of $\beta > 0$ will typically suffice.

Definition 2.2. *Let Φ_t be a member of the one-parameter group of diffeomorphism generated by the vector field $\nabla_\omega \mathcal{H}$.*[2] *Then, we call a measure μ invariant with respect to $\nabla_\omega \mathcal{H}$ if, for any measurable subset A of the phase space, we have*

$$\mu(\Phi_{-t}(A)) = \mu(A).$$

For our toy finite-dimensional Hamiltonian system, we can verify that the Gibbs measure on the phase space $\mathbb{R}^{2n}$ is invariant using conservation of the Hamiltonian and invariance of the Lebesgue measure.

Proposition 2.3. *The Gibbs measure μ is invariant under the flow of the Hamiltonian system given by (2.1).*

[2] In this case, $\Phi_t : (p(0), q(0)) \mapsto (p(t), q(t))$. In future sections, we will have $\Phi_t : u(0, \cdot) \mapsto u(t, \cdot)$.

Proof. Let μ be the Gibbs measure and $A \subset \mathbb{R}^{2n}$ be measurable. Then,

$$\begin{aligned}\mu(\Phi_{-t}(A)) &= \mu(\{(p(0), q(0)) \in \mathbb{R}^{2n} \mid (p(0), q(0)) \in \Phi_{-t}(A)\}) \\ &= \mu(\{(p(0), q(0)) \in \mathbb{R}^{2n} \mid (p(t), q(t)) \in A\}) \\ &= \frac{1}{Z}\int_A e^{-\mathcal{H}(p(t),q(t))}\, dp(t)\, dq(t) \\ &= \frac{1}{Z}\int_A e^{-\mathcal{H}(p(0),q(0))}\, dp(0)\, dq(0) \\ &= \mu(A),\end{aligned}$$

where we use that the Hamiltonian is conserved and the Lebesgue measure is invariant. □

3. Formal outline of the Gibbs measure

Now, we return to the nonlinear Schrödinger equation. [Tao06, Exercise 3.1] gives us that NLS is the formal Hamiltonian flow associated with the Hamiltonian functional $\mathcal{H} : H^s_x(\mathbb{T}^d) \to \mathbb{R}$ given for $u = u(t, \cdot)$ at time t by

$$\mathcal{H}(u) = \frac{1}{2}\int_{\mathbb{T}^d} |\nabla u|^2 + \frac{\lambda}{p+1}\int_{\mathbb{T}^d} |u|^{p+1}.$$

Then, we can rewrite NLS in terms of $\mathcal{H}$ as

$$\partial_t u = -i\frac{\partial \mathcal{H}}{\partial \overline{u}}.$$

As we did in the previous section, we would like to define a Gibbs measure μ by

$$d\mu = \frac{1}{Z}e^{-\mathcal{H}(u)}\, du.$$

Actually, we seek the similar

$$d\mu = \frac{1}{Z}e^{-\mathcal{H}(u)-\frac{1}{2}\mathcal{M}(u)}\, du,$$

where

$$\mathcal{M}(u) = \int_{\mathbb{T}^d} |u|^2$$

represents the mass, which is also conserved.[3]

In the previous section, $\mathcal{H}$ was a function on the finite-dimensional phase space $\mathbb{R}^{2n}$, so $dp\, dq$ was the well-defined Lebesgue measure on $\mathbb{R}^{2n}$. Here, du represents an analogue on the infinite-dimensional space $H^s_x(\mathbb{T}^d)$, but an infinite-dimensional Lebesgue measure does not exist. Therefore, this definition can only be made sense of formally. For a rigorous construction of the Gibbs measure, we must replace du by a tractable alternative.

[3] Adding $-\frac{1}{2}\mathcal{M}(u)$ in the exponent will replace a homogenous Sobolev $\dot{H}^s_x$-norm with its inhomogenous counterpart in the expression for $d\rho_s$. This will later translate to $\langle n\rangle^s$ in a denominator rather than $|n|^s$, avoiding an issue at $n = 0$.

We can proceed by formally rewriting $d\mu$ as

$$
\begin{aligned}
d\mu &= \frac{1}{Z} e^{-\mathcal{H}(u) - \frac{1}{2}\mathcal{M}(u)}\, du \\
&= \frac{1}{Z} e^{-\frac{\lambda}{p+1}\int_{\mathbb{T}^d} |u|^{p+1}} e^{-\frac{1}{2}\int_{\mathbb{T}^d} |u|^2 - \frac{1}{2}\int_{\mathbb{T}^d} |\nabla u|^2}\, du \\
&= \frac{1}{\widetilde{Z}} e^{-\frac{\lambda}{p+1}\int_{\mathbb{T}^d} |u|^{p+1}}\, d\rho_1,
\end{aligned}
$$

where

$$
d\rho_s = \frac{1}{Z_s} e^{-\frac{1}{2}\|u\|_{H_x^s}^2}\, du
$$

for $s \in \mathbb{R}$ is an infinite-dimensional analogue of the Gaussian measure and $\widetilde{Z}, Z_s$ are the respective normalization constants. Although du cannot be defined rigorously, $d\rho_s$ can. We can make sense of this construction of μ by defining ρ_s as the limit of finite-dimensional Gaussian measures.

The Fourier inversion formula allows us to write u as

$$
u(t,x) = \sum_{n \in \mathbb{Z}^d} \hat{u}(t,n) e^{in\cdot x},
$$

where $\hat{u}(t,n)$ is the Fourier series of $u(t,x)$ in x. Then, we can take the Littlewood-Paley projections $P_{\leq N}$ of u, which are given by the finite series

$$
u^N(t,x) := P_{\leq N} u(t,x) = \sum_{|n| \leq N} \hat{u}(t,n) e^{in\cdot x}. \tag{3.1}
$$

The following rigorous argument will construct invariant Gibbs measures for finite-dimensional truncations of NLS involving u^N, then take the limit as $N \to \infty$ to obtain a Gibbs measure for the genuine NLS.

4. Construction of the Gibbs measure

Motivated by our rigorous argument for finite-dimensional Hamiltonian systems and the formal convergence $u^N \to u$, we consider the finite-dimensional truncation of NLS,

$$
\begin{cases} i\partial_t u^N + \Delta u^N = \lambda P_{\leq N}(|u^N|^{p-1} u^N) \\ u^N(t_0, x) = P_{\leq N} u_0(x), \end{cases} \tag{4.1}
$$

which we call finite-dimensional because the low-frequency component $u^N = P_{\leq N} u$ reduces to a finite system of ODEs in the frequency space. In the sequel, we fix a time t and consider $u(x) = u(t,x)$, $u^N(x) = u^N(t,x)$ as functions of x.

For fixed $N \in 2^{\mathbb{N}}$, we construct the Gibbs measure μ for (4.1) as we did for the original NLS. Now, the infinite-dimensional analogue of the Gaussian measure ρ_s is replaced by

$$
d\rho_{s,N} = \frac{1}{Z_{s,N}} e^{-\frac{1}{2}\|u^N\|_{H_x^s}^2}\, du^N,
$$

which is a well-defined finite-dimensional Gaussian measure. Indeed, we can expand

$$\begin{aligned} d\rho_{s,N} &= \frac{1}{Z_{s,N}} e^{-\frac{1}{2}\|u^N\|_{H_x^s}^2}\, du^N \\ &= \frac{1}{Z_{s,N}} e^{-\frac{1}{2}\sum_{|n|\le N}\langle n\rangle^{2s}|\hat{u}(n)|^2} \prod_{|n|\le N} d\hat{u}(n) \\ &= \frac{1}{Z_{s,N}} \prod_{|n|\le N} e^{-\frac{1}{2}\langle n\rangle^{2s}|\hat{u}(n)|^2}\, d\hat{u}(n). \end{aligned}$$

For any $n \in \mathbb{Z}^d$, $d\hat{u}(n)$ represents the Lebesgue measure on $\mathbb{C}$, so each

$$d\nu_{s,n} := e^{-\frac{1}{2}\langle n\rangle^{2s}|\hat{u}(n)|^2}\, d\hat{u}(n) \tag{4.2}$$

is, after normalization, a Gaussian measure on $\mathbb{C}$ with mean zero and variance $2\langle n\rangle^{-2s}$. Therefore, $\rho_{s,N} = \frac{1}{Z_{s,N}}\prod_{|n|\le N} d\nu_{s,n}$ can be identified as a Gaussian measure on $\mathbb{C}^{(2N+1)^d}$.

4.1. **Probabilistic perspective.** It becomes easier to interpret this statement and take the limit $N \to \infty$ if we use probabilistic language. In particular, define the mutually independent complex Gaussian random variables $g_n : \Omega \to \mathbb{C}$ by $g_n(\omega) = \langle n\rangle^s \hat{u}(n)$ on some abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, (4.2) becomes

$$d\nu_{s,n} = e^{-\frac{1}{2}|g_n(\omega)|^2}\, d\frac{g_n(\omega)}{\langle n\rangle^s},$$

which is the push-forward measure $(g_n)_*\mathbb{P} = \mathbb{P} \circ g_n^{-1}$.

Now, (3.1) becomes

$$u^N(x) = \sum_{|n|\le N} \frac{g_n(\omega)}{\langle n\rangle^s} e^{in\cdot x},$$

which implies that we can take the limit $N \to \infty$ by replacing the finite sum with an infinite sum over all $n \in \mathbb{Z}^d$. Indeed, this will be our approach exactly.

First, we assert the following lemma about the distribution of the g_n's.

Lemma 4.1. *For each $n \in \mathbb{N}$, $\mathbb{E}[g_n] = 0$ and $\operatorname{Var}(g_n) = 2$.*

Proof. The density is an odd function, so $\mathbb{E}[g_n] = 0$. The variance can easily be computed as

$$\operatorname{Var}(g_n) = \mathbb{E}|g_n|^2 = \int_\Omega |g_n(\omega)|^2\, d\mathbb{P}(\omega) = 2.$$

□

Armed with the distribution of the g_n's, we proceed to take the limit as $N \to \infty$ and show that $u^N \to u$ in L^2_ω as $H^\sigma_x(\mathbb{T}^d)$-valued random variables for sufficiently small σ.

Proposition 4.2. *The sequence $(u^N)_N$ is a Cauchy sequence in $L^2_\omega(\Omega \to H^\sigma_x(\mathbb{T}^d))$ if and only if $\sigma < s - \frac{d}{2}$. In that case, it converges to u given by*

$$u(x) := \sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle^s} e^{in\cdot x}.$$

Proof. Let $M, N \in \mathbb{N}$ such that $M < N$. Then, for $\sigma \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\left[\left\|u^N - u^M\right\|^2_{H^\sigma_x}\right] &= \mathbb{E}\left[\sum_{M<|n|\le N} \frac{|g_n(\omega)|^2}{\langle n\rangle^{2s-2\sigma}}\right] \\
&= \sum_{M<|n|\le N} \frac{\mathbb{E}\,|g_n|^2}{\langle n\rangle^{2s-2\sigma}} \\
&= 2\sum_{M<|n|\le N} \frac{1}{\langle n\rangle^{2s-2\sigma}},
\end{aligned}$$

where we use Lemma 4.1 in the third step. The final expression converges to zero as $M, N \to \infty$ if and only if $2s - 2\sigma > d$, i.e. $\sigma < s - \frac{d}{2}$. In that case, $(u^N)_N$ is a Cauchy sequence. As $L^2_\omega(\Omega \to H^\sigma_x(\mathbb{T}^d))$ is complete, it must converge. We define u pointwise as the limit of the series. □

Now, we find that

$$d\rho_s = \frac{1}{Z_s} e^{-\frac{1}{2}\|u\|^2_{H^s_x}}\, du$$

is the induced probability measure on $H^\sigma_x(\mathbb{T}^d)$ for $\sigma < s - \frac{d}{2}$ under the map

$$\Omega \ni \omega \mapsto u(x) = \sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle^s} e^{in\cdot x} \in H^\sigma_x(\mathbb{T}^d).$$

That is, considering u as a random variable (i.e. as a function of ω),

$$\rho_s = u_*\mathbb{P} = \mathbb{P} \circ u^{-1}.$$

Heuristically, ρ_s can be interpreted as concentrating weight in regions of functions with small H^σ_x-norms. This intuition is made precise by the following tail estimate, which essentially says that $\|u\|_{H^\sigma_x}$ is a subgaussian random variable with respect to the probability measure ρ_s.

Proposition 4.3. *Let $\sigma < s - \frac{d}{2}$. Then, there exists a constant $c \in (0, \infty)$ such that*

$$\rho_s(\|u\|_{H^\sigma_x} > K) \lesssim e^{-cK^2}$$

for all $K > 0$.

Proof. We can compute

$$\begin{aligned}
\rho_s(\|u\|_{H^\sigma_x} > K) &= \rho_s(e^{c\|u\|^2_{H^\sigma_x}} > e^{cK^2}) \\
&= \int_{H^\sigma_x(\mathbb{T}^d)} \mathbb{1}_{\left\{\frac{\exp(c\|u\|^2)}{\exp(cK^2)} > 1\right\}}\, d\rho_s(u) \\
&\le e^{-cK^2} \int_{H^\sigma_x(\mathbb{T}^d)} e^{c\|u\|^2_{H^\sigma_x}}\, d\rho_s(u).
\end{aligned}$$

Using Plancherel's theorem,

$$\begin{aligned} C &:= \int_{H^\sigma_x(\mathbb{T}^d)} e^{c\|u\|^2_{H^\sigma_x}}\, d\rho_s(u) \\ &= \int_{H^\sigma_x(\mathbb{T}^d)} e^{c\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{2\sigma}|\hat{u}(n)|^2}\, d\rho_s(u) \\ &= \int_{H^\sigma_x(\mathbb{T}^d)} \prod_{n\in\mathbb{Z}^d} e^{c\langle n\rangle^{2\sigma}|\hat{u}(n)|^2}\, d\rho_s(u). \end{aligned}$$

As $g_n(\omega) = \langle n\rangle^s \hat{u}(n)$ and $(g_n)_n$ is a collection of mutually independent random variables, this is equal (up to a constant) to

$$\int_{\mathbb{C}} \prod_{n\in\mathbb{Z}^d} e^{c\langle n\rangle^{2\sigma-2s}|g_n|^2} e^{-\frac{1}{2}|g_n|^2}\, dg_n = \prod_{n\in\mathbb{Z}^d} \int_{\mathbb{C}} e^{c\langle n\rangle^{2\sigma-2s}|g_n|^2} e^{-\frac{1}{2}|g_n|^2}\, dg_n,$$

which can be integrated to obtain (again up to a constant)

$$\prod_{n\in\mathbb{Z}^d} \frac{1}{1-2c\langle n\rangle^{2\sigma-2s}} = \prod_{n\in\mathbb{Z}^d}\left(1+\frac{2c\langle n\rangle^{2\sigma-2s}}{1-2c\langle n\rangle^{2\sigma-2s}}\right),$$

which is finite if and only if

$$\sum_{n\in\mathbb{Z}^d} \frac{2c\langle n\rangle^{2\sigma-2s}}{1-2c\langle n\rangle^{2\sigma-2s}} < \infty.$$

We have that $\sigma < s - \frac{d}{2}$, so $2s - 2\sigma > d$, so the series converges. Therefore, $C \in (0,\infty)$, so

$$\rho_s(\|u\|_{H^\sigma_x} > K) \lesssim e^{-cK^2}$$

as desired. □

4.2. **Construction of Gibbs measure.** Now that we have ρ_s for all $s \in \mathbb{R}$, μ can be defined on $H^\sigma_x(\mathbb{T}^d)$ for $\sigma < 1 - \frac{d}{2}$ by

$$d\mu = \frac{1}{\widetilde{Z}} e^{-\frac{\lambda}{p+1}\int_{\mathbb{T}^d}|u|^{p+1}}\, d\rho_1. \tag{4.3}$$

Going forward, we restrict ourselves to the one-dimensional torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.[4] We would like to verify that μ is a nontrivial probability measure on $H^\sigma_x(\mathbb{T})$ even after we take the limit $N \to \infty$. This requires the density

$$\frac{d\mu}{d\rho_1} \propto e^{-\frac{\lambda}{p+1}\int_{\mathbb{T}}|u|^{p+1}}$$

to be integrable with respect to ρ_1. In the defocusing case (where $\lambda = 1$), this is easily verified using the Sobolev embedding theorem.

Theorem 4.4. *In the defocusing case, μ is a nontrivial probability measure on $H^\sigma_x(\mathbb{T})$ for $\sigma < \frac{1}{2}$.*

Proof. Suppose that $\sigma < \frac{1}{2}$. Using Proposition 4.2, $u^N \to u$ in $L^2_\omega(\Omega \to H^\sigma_x(\mathbb{T}))$, so ρ_1 is well-defined as the probability measure on $H^\sigma_x(\mathbb{T})$ induced by the map $\omega \mapsto u$.

[4]The case $d \geq 2$ is substantially different, as μ would be defined on $H^\sigma_x(\mathbb{T}^d)$ only for $\sigma < 0$.

By the Sobolev embedding theorem, for any $\epsilon > 0$ sufficiently small,

$$\int_{\mathbb{T}} |u|^{p+1} = \|u\|_{L_x^{p+1}}^{p+1} \lesssim \|u\|_{H_x^{\frac{1}{2}-\epsilon}}^{p+1} < \infty$$

almost surely. Therefore,

$$0 < e^{-\frac{1}{p+1}\int_{\mathbb{T}}|u|^{p+1}} \leq 1$$

almost surely, so μ is a nontrivial probability measure on $H_x^\sigma(\mathbb{T})$ for $\sigma < \frac{1}{2}$. □

4.3. **Focusing case.** Unfortunately, the focusing case (where $\lambda = -1$) is more difficult to approach. Let $p > 1$. We may compute

$$\int_{\mathbb{T}} |u|^{p+1} = \|u\|_{L_x^{p+1}}^{p+1} \geq \|u\|_{L_x^2}^{p+1} \asymp \|\hat{u}\|_{\ell_n^2}^{p+1} = \left[\sum_{n\in\mathbb{Z}} |\hat{u}(n)|^2\right]^{\frac{p+1}{2}} \geq \sum_{n\in\mathbb{Z}} |\hat{u}(n)|^{p+1} = \sum_{n\in\mathbb{Z}} \left|\frac{g_n(\omega)}{\langle n\rangle}\right|^{p+1},$$

where we use that $L^{p+1}(\mathbb{T}) \subset L^2(\mathbb{T})$ (as $p + 1 > 2$), Plancherel's theorem, and convexity of $|\cdot|^{\frac{p+1}{2}}$.

Therefore,

$$\int_{H_x^\sigma(\mathbb{T})} e^{\frac{1}{p+1}\int_{\mathbb{T}}|u|^{p+1}}\, d\rho_1 \geq \prod_{n\in\mathbb{Z}} \mathbb{E}\left[e^{\frac{1}{p+1}\left|\frac{g_n(\omega)}{\langle n\rangle}\right|^{p+1}}\right] = \infty,$$

where we use that each factor is infinite because $p + 1 > 2$. We find that the density

$$\frac{d\mu}{d\rho_1} \propto e^{\frac{1}{p+1}\int_{\mathbb{T}}|u|^{p+1}}$$

is not integrable, so μ fails to be a well-defined probability measure in the focusing case.

The solution is to introduce a cutoff of the L^2-norm (representing the mass), replacing (4.3) with

$$d\mu = \frac{1}{\widetilde{Z}} \mathbb{1}_{\{\|u\|_{L^2} \leq r\}} e^{\frac{1}{p+1}\int_{\mathbb{T}}|u|^{p+1}}\, d\rho_1$$

for some $r > 0$. As proved in [LRS88, Theorem 2.1] and [Bou94, Lemma 3.10], this L^2-truncated density is integrable and yields a well-defined Gibbs measure when $1 < p < 5$ for any $r > 0$.[5] Unfortunately, the L^2-norm cutoff does not suffice for $p > 5$.

5. Invariance of the Gibbs measure

This construction of Gibbs measures allows for low-regularity global well-posedness results even in the periodic setting. For example, we have the following probabilistic result from [Bou94, Lemma 4.4].

Theorem 5.1. *Let $p \leq 5$. NLS is, almost surely with respect to the Gibbs measure μ, globally well-posed on $H_x^\sigma(\mathbb{T})$ for $\sigma < \frac{1}{2}$.*

Based on this almost sure global well-posedness result, [Bou94, p. 17] also proves invariance analogous to Proposition 2.3.

Theorem 5.2. *For $p \in [3, 5]$, the Gibbs measure μ is invariant under the flow of NLS in one dimension.*

[5] For sufficiently small r, the density is integrable even in the mass-critical $p = 5$ case, so called because the scaling-critical regularity is $s_c = 0$.

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