Martingales in Banach Spaces and the Randomized UMD Properties

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Abstract

First introduced in [Mau75] before being investigated at length in [Bur81], UMD spaces (Banach spaces for which martingale difference sequences converge unconditionally) play a central role in the modern theory of Banach space-valued stochastic and harmonic analysis. In this paper, we systematically develop the framework for the randomized analogues introduced in [Gar90]: UMD^+ and UMD^- spaces, which arise by replacing deterministic signs with Rademacher sequences in martingale inequalities. We discuss the general theory of martingales in Banach spaces (including several important inequalities), then treat the randomized UMD properties as Banach space properties in their own right. Our study focuses on geometric consequences for K-convexity, type, cotype, and reflexivity. The results of our exploration underscore the rich interplay between probability, analysis, and geometry in the study of Banach spaces.

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Chapter 1

Introduction and Notation

1.1 Introduction

There exists a rich theory of classical analysis involving measure and integration, function spaces, and the convergence of sequences of functions. This theory has found numerous applications across almost all areas of pure and applied mathematics: to the study of ordinary and partial differential equations, probability theory and stochastic analysis, Fourier and harmonic analysis, as well as across the disciplines of physics, statistics, and engineering.

In each of these applications, mathematical questions typically arise about the existence, uniqueness, and properties of measurable functions from a measure space S to a scalar field $\mathbb{R}$ or $\mathbb{C}$. In the context of probability theory, such functions are called $\mathbb{R}$ or $\mathbb{C}$-valued random variables and form the basis for almost the entirety of our study of probability and stochastic processes. One of the most basic classes of stochastic processes—indexed families of random variables—is martingales.

The modern study of scalar-valued martingales began with Doob's publication in 1953 of the first edition of [Doo90]. In this paper, we are principally concerned with the study of martingales which take values in more general Banach spaces than $\mathbb{R}$ or $\mathbb{C}$, a theory which began in [Cha64]. Although seemingly purely probabilistic objects, such martingales are actually immensely useful tools for characterizing the analytical and geometric properties of the Banach spaces in which they take values. This will be the main theme of our paper: probabilistic statements about martingales actually have many analytical and geometric consequences.

Our study will focus on refinements and extensions of a particular class of Banach spaces called UMD spaces (for "unconditionality of martingale differences"). First introduced in [Mau75, Pis75], they were connected to the geometry of Banach spaces in [Bur81] and Banach space-valued harmonic analysis in [Bur83]. The essence of the theory is that Banach spaces in which the difference sequences of martingales converge unconditionally also have many other analytical and geometric properties which make them excellent settings for an extension of the classical theory of analysis. Instead of considering functions from a measure space S to a scalar field $\mathbb{R}$ or $\mathbb{C}$, we will study functions from S to a UMD space X (and later Banach spaces X which are not quite UMD, but slightly generalize the UMD property).

Let us begin with the definition of a UMD space.

Definition 1.1.1 (UMD)**.** *For $p \in (1, \infty)$, a Banach space X is called a* UMD$_p$ space *if there exists a constant $\beta \in (0, \infty)$ such that for any X-valued L^p-martingale difference sequence $(df_n)_{n=1}^N$ on a σ-finite measure space $(S, \mathcal{A}, \mu)$ and scalars $|\epsilon_n| = 1$, $n = 1, \ldots, N$, we have*

$$\left\| \sum_{n=1}^{N} \epsilon_n df_n \right\|_{L^p(S;X)} \leq \beta \left\| \sum_{n=1}^{N} df_n \right\|_{L^p(S;X)}.$$

Although the definition appears to depend on the choice of p, it is a deep result of [Mau75] and [Bur81] that this condition is satisfied for all $p \in (1, \infty)$ if it is satisfied by some.

Theorem 1.1.2. *For $p, q \in (1, \infty)$, X is UMD$_p$ if and only if it is UMD$_q$.*

In this case, we simply say that X is a UMD space or that X satisfies the UMD property.

Although the UMD property is at first glance a probabilistic statement about martingales, there is also a harmonic analytical interpretation. For a Banach space X, let us formally define the Hilbert transform on $L^p(\mathbb{R}; X)$ as (the principal value of)

$$\text{"}Hf(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x - y} \, \mathrm{d}y.\text{"}$$

It is natural to ask whether H is a bounded operator on $L^p(\mathbb{R}; X)$. Two deep results of Burkholder and Bourgain give us that H is a bounded operator on $L^p(\mathbb{R}; X)$ if and only if X is a UMD space. The forward direction is due to [Bou83], while the backward direction is given by [Bur83]. See [Pis16, Chapter 6] for extensive discussion of this equivalence.

There is also a *more* probabilistic perspective that appears when we replace the deterministic unit norm scalars with Rademacher random variables (which are uniformly distributed over the unit circle). The following randomized UMD inequalities were initially proved in [Gar86] and can be used to establish a quantitative relationship between UMD constants and the operator norm of the Hilbert transform.

Theorem 1.1.3 (Randomized UMD inequalities)**.** *A Banach space X is a UMD space if and only if for all $p \in (1,\infty)$, there exist constants $\beta^{\pm} \in (0,\infty)$ such that for any X-valued L^p-martingale difference sequence $(df_n)_{n=1}^N$ on a σ-finite measure space $(S,\mathcal{A},\mu)$, we have*

$$\frac{1}{\beta^-}\left\|\sum_{n=1}^N df_n\right\|_{L^p(S;X)} \leq \left\|\sum_{n=1}^N \varepsilon_n df_n\right\|_{L^p(S\times\Omega;X)} \leq \beta^+\left\|\sum_{n=1}^N df_n\right\|_{L^p(S;X)},$$

where $(\varepsilon_n)_{n=1}^N$ is a Rademacher sequence on a probability space Ω.

Theorem 1.1.3 motivates the following definitions for generalizations of UMD spaces—each requiring one of the two inequalities—which were first isolated and examined in their own right in [Gar90].

Definition 1.1.4 (UMD^+)**.** *For $p \in (1,\infty)$, a Banach space X is called a* UMD_p^+ space *if there exists a constant $\beta^+ \in (0,\infty)$ such that for any X-valued L^p-martingale difference sequence $(df_n)_{n=1}^N$ on a σ-finite measure space $(S,\mathcal{A},\mu)$, we have*

$$\left\|\sum_{n=1}^N \varepsilon_n df_n\right\|_{L^p(S\times\Omega;X)} \leq \beta^+\left\|\sum_{n=1}^N df_n\right\|_{L^p(S;X)}$$

where $(\varepsilon_n)_{n=1}^N$ is a Rademacher sequence on a probability space Ω.

Definition 1.1.5 (UMD^-)**.** *For $p \in (1,\infty)$, a Banach space X is called a* UMD_p^- space *if there exists a constant $\beta^- \in (0,\infty)$ such that for any X-valued L^p-martingale difference sequence $(df_n)_{n=1}^N$ on a σ-finite measure space $(S,\mathcal{A},\mu)$, we have*

$$\left\|\sum_{n=1}^N df_n\right\|_{L^p(S;X)} \leq \beta^-\left\|\sum_{n=1}^N \varepsilon_n df_n\right\|_{L^p(S\times\Omega;X)}$$

where $(\varepsilon_n)_{n=1}^N$ is a Rademacher sequence on a probability space Ω.

Analogous to Theorem 1.1.2, the following result of [Gar90] shows that the UMD$^+$ and UMD$^-$ properties hold irrespective of one's choice of $p \in (1, \infty)$.

Theorem 1.1.6. *For $p, q \in (1, \infty)$, X is UMD$_p^+$ (resp. UMD$_p^-$) if and only if it is UMD$_q^+$ (resp. UMD$_q^-$).*

In this case, we simply say that X is a UMD$^+$ or UMD$^-$ space. We will refer to the UMD$^+$ and UMD$^-$ properties together as the *randomized UMD properties* in recognition of the roles that Rademacher random variables play in their definitions.

With these definitions and Theorem 1.1.6, we can now rephrase Theorem 1.1.3 as stating that a Banach space X is UMD if and only if it is both UMD$^+$ and UMD$^-$.

The UMD$^-$ property is strictly weaker than the UMD property (Example 3.3.3 shows that ℓ^1 is UMD$^-$ but not UMD$^+$ and thus not UMD). It remains an open problem whether the same is true for UMD$^+$ or whether it is equivalent to the UMD property. At the very least, [Gei99, Corollary 5] shows that there is not a general linear bound between the optimal UMD and UMD$^+$ constants for arbitrary X, which hints that this conjecture may be false. One possible avenue for proving the affirmative may be showing that the UMD$^+$ property is equivalent to the boundedness of the Hilbert transform.

There is a wealth of literature examining UMD and randomized UMD spaces with examples of such spaces. Existing literature focuses on either consequences of the UMD property or particular examples of spaces which satisfy some or none of the UMD or randomized UMD properties. Textbooks such as [Pis16] and [HvVW16] also discuss these properties, examples, and the general use of UMD spaces as a convenient setting for Banach space-valued analysis.

At the moment, there are no surveys which study the randomized UMD properties other than as consequences of the UMD property via Theorem 1.1.3. As many convenient properties of UMD spaces are inherited by UMD$^+$ spaces, we believe it useful to study UMD$^+$ and UMD$^-$ as Banach space properties in their own right rather than as corollaries of the UMD property.

In this paper, we will discuss the consequences of the randomized UMD properties with a focus on treating them independently of one another and the UMD property. Chapter 2 concerns the general theory of martingales in Banach spaces, particularly the inequalities of Doob, Kahane, and Kahane-Khintchine, which will be useful for us in later chapters. Chapter 3 discusses the basic properties and examples of UMD$^+$ and UMD$^-$ spaces, with an eye for those that are different from UMD spaces. Chapter 4 discusses the consequences of the UMD$^+$ and UMD$^-$ properties for K-convexity, type, cotype, and other geometrical notions. Finally, Chapter 5 concludes with a discussion of the remaining open questions.

1.2 Notation

The symbol $\mathbb{K}$ is used to stand in for the underlying scalar field of a Banach space, either $\mathbb{R}$ or $\mathbb{C}$. For the unit circle of $\mathbb{K}$, we write $S_{\mathbb{K}} = \{z \in \mathbb{K} \mid |z| = 1\}$.

We use $(S, \mathcal{A}, \mu)$ and $(T, \mathcal{B}, \nu)$ to denote σ-finite measure spaces, typically those on which a martingale is defined. Generally, $(f_n)_{n=1}^N$ is a martingale defined on S and $(df_n)_{n=1}^N$ is its difference sequence.

We use $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ to denote probability spaces. We also use $\mathbb{E}[\cdot]$ and $\widetilde{\mathbb{E}}[\cdot]$ to denote the expectation with respect to random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, respectively.

Unless otherwise specified, $(\varepsilon_n)_{n=1}^N$ is an independent Rademacher sequence on $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. a sequence which is independent and uniformly distributed along $S_{\mathbb{K}}$.[1] The same holds for $(\widetilde{\varepsilon}_n)_{n=1}^N$ on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$. We will distinguish real Rademacher sequences (those which are independent and uniformly distributed along $\{-1, 1\}$) that take values in complex Banach spaces by writing $(r_n)_{n=1}^N$ rather than $(\varepsilon_n)_{n=1}^N$.

For $p \in [1, \infty]$, we denote by p' the Hölder conjugate of p: the unique $p' \in [1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

with the convention that $1/\infty = 0$.

[1] When $\mathbb{K} = \mathbb{C}$, these are sometimes known in the literature as *Steinhaus* random variables and sequences, but we will not use this terminology.

Chapter 2

Martingales in Banach Spaces

Before we treat the UMD and randomized UMD properties, we must generalize the machinery of martingales to Banach spaces. We begin this chapter with the construction of conditional expectations for Banach space-valued random variables. In Section 2.2, we define Banach space-valued martingales and prove Doob's maximal inequalities. Finally, in Section 2.3, we discuss Rademacher sequences before proving Kahane's contraction principle and the Kahane-Khintchine inequalities. Each of these theorems will be immensely useful to us during our study of the randomized UMD properties in the next chapter.

2.1 Conditional expectation

The goal of this section is to generalize the notion of conditional expectation to the Banach space setting, Generally, conditional expectations are defined for functions on probability spaces (i.e. random variables). We will choose to define conditional expectations for a broader class of functions defined on σ-finite measure spaces, which is convenient for harmonic analysis set on UMD spaces. These arguments are classical, but our presentation is based mainly on that of [HvVW16].

For all of this section, let $(S, \mathcal{A}, \mu)$ be a σ-finite measure space and let $\mathcal{F} \subseteq \mathcal{A}$ be a sub-σ-algebra for which μ is σ-finite. For any function $f \in L^0(S; X)$, write

$$\mathcal{F}_f := \{F \in \mathcal{F} \mid \mathbb{1}_F f \in L^1(S; X)\}$$

for the sets in $\mathcal{F}$ over which f is integrable. Then, we define the following class of functions for which we will construct conditional expectations.

Definition 2.1.1 (σ-integrable)**.** *For a σ-finite sub-σ-algebra $\mathcal{F} \subseteq \mathcal{A}$, a function $f \in L^0(S;X)$ is called* σ-integrable over $\mathcal{F}$ *if S can be covered by a countable collection $(F_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{F}_f$.*

We shall call such a countable collection an *exhausting sequence* for f in $\mathcal{F}$. If $(F_n)_{n\in\mathbb{N}}$ is an exhausting sequence, $(G_n)_{n\in\mathbb{N}}$ given by $G_1 = F_1$ and $G_n = F_n \setminus \cup_{j=1}^{n-1} F_n$ is also an exhausting sequence, so we can assume without loss of generality that any exhausting sequence is pairwise disjoint, Now, if f is σ-integrable over $\mathcal{F}$, we can split it up into countably many parts with disjoint support which are each integrable.

For such functions, our goal is to construct conditional expectations which satisfy the following definition.

Definition 2.1.2 (Conditional expectation)**.** *Let $f \in L^0(S;X)$ be σ-integrable over $\mathcal{F}$. A function $g \in L^0(S,\mathcal{F};X)$ is called a* conditional expectation with respect to $\mathcal{F}$ *of f if*

$$\int_F g \, \mathrm{d}\mu = \int_F f \, \mathrm{d}\mu \tag{2.1.1}$$

for all $F \in \mathcal{F}_f \cap \mathcal{F}_g$.

Without proof, we assume the existence and uniqueness of conditional expectation in the scalar case as well as standard properties such as conditional Jensen's inequality and L^p-contractivity. The details will be in most probability theory references, such as [Dur10]. Now, we will prove existence and uniqueness in the Banach space case.

2.1.1 Uniqueness of conditional expectation

In order to prove existence and uniqueness, we would like to replace eq. (2.1.1) with another condition which is easier to verify. For that reason, we define an *exhausting ideal* for $\mathcal{F}$ as a subset $\mathcal{C} \subseteq \mathcal{F}$ which is an ideal in the sense that $C \cap F \in \mathcal{C}$ for any $C \in \mathcal{C}, F \in \mathcal{F}$ and for which S can be covered by a countable subcollection of sets in $\mathcal{C}$. It is easy to check that any exhausting ideal is closed under countable intersections and contains a countable pairwise disjoint cover of S.

If $f \in L^0(S,\mathcal{F};X)$ is σ-integrable, then for an exhausting sequence $(F_n)_{n\in\mathbb{N}}$, f is integrable on $F_n \cap \{\|f\|_X \le m\}$ for all $m, n \in \mathbb{N}$, so $\{F_n \cap \{\|f\|_X \le m\}\}_{n,m\in\mathbb{N}}$ is also an exhausting sequence for f in $\mathcal{F}$. It follows that $\mathcal{F}_f$ is an exhausting ideal for $\mathcal{F}$. Now, we show that we can verify that a function $g \in L^0(S,\mathcal{F};X)$ is a conditional expectation of $f \in L^0(S;X)$ with respect to $\mathcal{F}$ by checking only sets in an exhausting ideal for $\mathcal{F}_f \cap \mathcal{F}_g$.

Lemma 2.1.3. *For $f \in L^0(S;X)$ which is σ-integrable over $\mathcal{F}$, $g \in L^0(S,\mathcal{F};X)$ is a conditional expectation of f with respect to $\mathcal{F}$ if and only if there is an exhausting ideal $\mathcal{C} \subseteq \mathcal{F}_f \cap \mathcal{F}_g$ for $\mathcal{F}$ such that*

$$\int_C g \,\mathrm{d}\mu = \int_C f \,\mathrm{d}\mu$$

for all $C \in \mathcal{C}$.

Proof. If g is a conditional expectation of f with respect to $\mathcal{F}$, $\mathcal{F}_f$ and $\mathcal{F}_g$ are exhausting ideals for $\mathcal{F}$ by the discussion above this lemma, so their intersection is as well. Then, the definition of conditional expectation means that the desired identity holds for all $C \in \mathcal{F}_f \cap \mathcal{F}_g$.

Suppose instead that the identity holds for all $C \in \mathcal{C}$ where $\mathcal{C} \subseteq \mathcal{F}_f \cap \mathcal{F}_g$ is an exhausting ideal. Fix $F \in \mathcal{F}_f \cap \mathcal{F}_g$ and cover S by a pairwise disjoint subcollection $(C_n)_{n\in\mathbb{N}}$ of $\mathcal{C}$. Then, $F \cap C_n \in \mathcal{C}$ for $n \in \mathbb{N}$, so

$$\int_{F\cap C_n} g \,\mathrm{d}\mu = \int_{F\cap C_n} f \,\mathrm{d}\mu.$$

Summing over all $n \in \mathbb{N}$ and using the dominated convergence theorem, it follows that g is a conditional expectation of f with respect to $\mathcal{F}$. □

This lemma allows us to verify and compare conditional expectations using only sets in an exhausting ideal $\mathcal{C} \subseteq \mathcal{F}_f$. We use this tool to prove two lemmas for the scalar case from which uniqueness of conditional expectation will follow. First, we show that comparing the integral over sets in an exhausting ideal with zero allows us to compare the integrand with zero.

Lemma 2.1.4. *Let $f \in L^0(S,\mathcal{F})$ and suppose that $\mathcal{C} \subseteq \mathcal{F}_f$ is an exhausting ideal for $\mathcal{F}$. If $\int_C f \,\mathrm{d}\mu \geq 0$ (resp. $\leq 0, = 0$) for all $C \in \mathcal{C}$, then $f \geq 0$ (resp. $\leq 0, = 0$) almost everywhere.*

Proof. As f is $\mathcal{F}$-measurable, $F_k := \{f < -\frac{1}{k}\} \in \mathcal{F}$ for all $k \geq 1$. Fix a set $C \in \mathcal{C}$. Then, $C \cap F_k \in \mathcal{C}$, which implies that $\mu(C \cap F_k) < \infty$ (otherwise f would not be integrable on the set) and

$$0 \leq \int_{C\cap F_k} f \,\mathrm{d}\mu \leq -\frac{1}{k}\mu(C \cap F_k) \leq 0,$$

so $\mu(C \cap F_k) = 0$. We have that $\{f < 0\} = \cup_{k \geq 1} F_k$, so it follows that $\mu(C \cap \{f < 0\}) = 0$. As an exhausting ideal, $\mathcal{C}$ contains a cover of S, so applying this identity to all sets in the cover, we find that $\mu(\{f < 0\}) = 0$, so $f \geq 0$ almost everywhere. Apply this fact with $-f$ for "$f \leq 0$" and combine the two cases for "$f = 0$". $\square$

In the second lemma, we show that, for a conditional expectation $g \in L^0(S)$ of a scalar-valued $f \in L^0(S)$, $\mathcal{F}_f \subseteq \mathcal{F}_g$ and their integrals over sets in $\mathcal{F}_f$ agree.

Lemma 2.1.5. *Let $f \in L^0(S)$ be σ-integrable over $\mathcal{F}$. Then, $\mathcal{F}_f \subseteq \mathcal{F}_{\mathbb{E}[f|\mathcal{F}]}$ and*

$$\int_F \mathbb{E}[f \mid \mathcal{F}] \, \mathrm{d}\mu = \int_F f \, \mathrm{d}\mu$$

for all $F \in \mathcal{F}_f$.

Proof. We will prove this lemma for the real-valued case, from which the complex-valued case follows using that $\mathcal{F}_f = \mathcal{F}_{\operatorname{Re} f} \cap \mathcal{F}_{\operatorname{Im} f}$ and $\mathcal{F}_{\mathbb{E}[f|\mathcal{F}]} = \mathcal{F}_{\operatorname{Re} \mathbb{E}[f|\mathcal{F}]} \cap \mathcal{F}_{\operatorname{Im} \mathbb{E}[f|\mathcal{F}]}$, so that $\operatorname{Re} \mathbb{E}[f \mid \mathcal{F}]$ and $\operatorname{Im} \mathbb{E}[f \mid \mathcal{F}]$ are the conditional expectations of $\operatorname{Re} f$ and $\operatorname{Im} f$.

By Lemma 2.1.3, $\mathcal{C} = \mathcal{F}_f \cap \mathcal{F}_{\mathbb{E}[f|\mathcal{F}]}$ is an exhausting ideal for $\mathcal{F}$. Therefore, there exists a pairwise disjoint cover $(C_n)_{n \in \mathbb{N}}$ of S in $\mathcal{C}$. Fix a set $F \in \mathcal{F}_f$. As $\mathbb{E}[f \mid \mathcal{F}]$ is $\mathcal{F}$-measurable, $\{\mathbb{E}[f \mid \mathcal{F}] \geq 0\} \in \mathcal{F}$, so $F \cap C_n \cap \{\mathbb{E}[f \mid \mathcal{F}] \geq 0\} \in \mathcal{C}$ and

$$\int_{F \cap C_n \cap \{\mathbb{E}[f|\mathcal{F}] \geq 0\}} \mathbb{E}[f \mid \mathcal{F}] \, \mathrm{d}\mu = \int_{F \cap C_n \cap \{\mathbb{E}[f|\mathcal{F}] \geq 0\}} f \, \mathrm{d}\mu.$$

Summing over n, then using the monotone convergence theorem on the left-hand side and the dominated convergence theorem on the right-hand side,

$$\int_{F \cap \{\mathbb{E}[f|\mathcal{F}] \geq 0\}} \mathbb{E}[f \mid \mathcal{F}] \, \mathrm{d}\mu = \int_{F \cap \{\mathbb{E}[f|\mathcal{F}] \geq 0\}} f \, \mathrm{d}\mu.$$

The same argument holds for the integrals over $F \cap \{\mathbb{E}[f \mid \mathcal{F}] < 0\}$, so

$$\begin{aligned}
\int_F |\mathbb{E}[f \mid \mathcal{F}]| \, \mathrm{d}\mu &= \int_{F \cap \{\mathbb{E}[f|\mathcal{F}] \geq 0\}} \mathbb{E}[f \mid \mathcal{F}] \, \mathrm{d}\mu - \int_{F \cap \{\mathbb{E}[f|\mathcal{F}] < 0\}} \mathbb{E}[f \mid \mathcal{F}] \, \mathrm{d}\mu \\
&= \int_{F \cap \{\mathbb{E}[f|\mathcal{F}] \geq 0\}} f \, \mathrm{d}\mu - \int_{F \cap \{\mathbb{E}[f|\mathcal{F}] < 0\}} f \, \mathrm{d}\mu \\
&\leq \int_F |f| \, \mathrm{d}\mu \\
&< \infty,
\end{aligned}$$

which implies that $F \in \mathcal{F}_{\mathbb{E}[f|\mathcal{F}]}$. Also,

$$\begin{aligned}\int_F \mathbb{E}[f \mid \mathcal{F}] \,\mathrm{d}\mu &= \int_{F\cap\{\mathbb{E}[f|\mathcal{F}]\geq 0\}} \mathbb{E}[f \mid \mathcal{F}] \,\mathrm{d}\mu + \int_{F\cap\{\mathbb{E}[f|\mathcal{F}]<0\}} \mathbb{E}[f \mid \mathcal{F}] \,\mathrm{d}\mu \\ &= \int_{F\cap\{\mathbb{E}[f|\mathcal{F}]\geq 0\}} f \,\mathrm{d}\mu + \int_{F\cap\{\mathbb{E}[f|\mathcal{F}]<0\}} f \,\mathrm{d}\mu \\ &= \int_F f \,\mathrm{d}\mu,\end{aligned}$$

as desired. □

With these lemmas for scalar conditional expectation, we are now prepared to prove uniqueness of conditional expectation in the Banach space-valued case.

Theorem 2.1.6 (Uniqueness of conditional expectation)**.** *Suppose that $f \in L^0(S; X)$ is σ-integrable over a σ-finite sub-σ-algebra $\mathcal{F} \subseteq \mathcal{A}$. If $g, h \in L^0(S; X)$ are conditional expectations of f with respect to $\mathcal{F}$, then $g = h$ almost everywhere.*

Proof. For any $x^* \in X^*$, $\langle g, x^*\rangle$ and $\langle h, x^*\rangle$ are conditional expectations for $\langle f, x^*\rangle$. By Lemma 2.1.5, for any $F \in \mathcal{F}_{\langle f,x^*\rangle}$, we have that $F \in \mathcal{F}_{\langle g,x^*\rangle} \cap \mathcal{F}_{\langle h,x^*\rangle}$ and

$$\int_F \langle g, x^*\rangle \,\mathrm{d}\mu = \int_F \langle f, x^*\rangle \,\mathrm{d}\mu = \int_F \langle h, x^*\rangle \,\mathrm{d}\mu.$$

Applying Lemma 2.1.4 to $\langle g - h, x^*\rangle$ and $\mathcal{C} = \mathcal{F}_{\langle f,x^*\rangle}$, this implies that $\langle g, x^*\rangle = \langle h, x^*\rangle$ almost everywhere. Our choice of $x^* \in X^*$ was arbitrary, so this holds for all such x^*, so it follows that $g = h$ almost everywhere. □

Now that we have shown uniqueness, we will denote by $\mathbb{E}[f \mid \mathcal{F}]$ the unique conditional expectation of $f \in L^0(S; X)$ with respect to $\mathcal{F}$, if it exists.

2.1.2 Existence of conditional expectation

It remains to prove the existence of the conditional expectation $\mathbb{E}[f \mid \mathcal{F}]$ for any $f \in L^0(S; X)$ which is σ-integrable over $\mathcal{F}$. Recall from the scalar case that the conditional expectation operator $\mathbb{E}[\cdot \mid \mathcal{F}]$ is defined on $L^2(S)$ as the orthogonal projection onto $L^2(S, \mathcal{F})$, which extends to a contractive operator on $L^1(S)$ satisfying

$$\int_F \mathbb{E}[f \mid \mathcal{F}] \,\mathrm{d}\mu = \int_F f \,\mathrm{d}\mu$$

for $F \in \mathcal{F}$.

In order to extend this operator to the Banach space case, we employ the following lemma, which gives that certain operators on $L^p(S)$ spaces can be extended to $L^p(S;X)$ spaces.

Lemma 2.1.7. *Fix $p \in [1,\infty)$ and $q \in [1,\infty]$. Let $(S,\mathcal{A},\mu)$ and $(T,\mathcal{B},\nu)$ be σ-finite measure spaces, let $E : L^p(S) \to L^q(T)$ be a bounded linear operator, and let X be a Banach space. If $Ef \geq 0$ for all $f \geq 0$, then $E \otimes I_X$ extends uniquely to a bounded operator from $L^p(S;X)$ to $L^q(T;X)$, which satisfies*

$$\|E \otimes I_X\|_{L^p(S;X)\to L^q(T;X)} = \|E\|_{L^p(S)\to L^q(T)}.$$

Proof. Let $f \in L^p(S) \otimes X$ be a simple function. We can write

$$f = \sum_{n=1}^{N} \mathbb{1}_{A_n} \otimes x_n$$

for $A_1,\dots,A_N$ pairwise disjoint. As $Ef \geq 0$ for any $f \geq 0$, $|E\mathbb{1}_{A_n}| = E\mathbb{1}_{A_n}$. Therefore,

$$\begin{aligned}
\left\|(E\otimes I_X)\left(\sum_{n=1}^{N} \mathbb{1}_{A_n}\otimes x_n\right)\right\|_{L^q(T;X)} &= \left(\int_T \left\|\sum_{n=1}^{N} E\mathbb{1}_{A_n}\otimes x_n\right\|^q \mathrm{d}\nu\right)^{\frac{1}{q}} \\
&\leq \left(\int_T \left(\sum_{n=1}^{N} |E\mathbb{1}_{A_n}|\|x_n\|_X\right)^q \mathrm{d}\nu\right)^{\frac{1}{q}} \\
&= \left(\int_T \left(E\sum_{n=1}^{N} \mathbb{1}_{A_n}\|x_n\|_X\right)^q \mathrm{d}\nu\right)^{\frac{1}{q}} \\
&= \left\|E\sum_{n=1}^{N} \mathbb{1}_{A_n}\|x_n\|_X\right\|_{L^q(T)} \\
&\leq \|E\|_{L^p(S)\to L^q(T)}\left\|\sum_{n=1}^{N} \mathbb{1}_{A_n}\|x_n\|_X\right\|_{L^p(S)} \\
&= \|E\|_{L^p(S)\to L^q(T)}\left\|\sum_{n=1}^{N} \mathbb{1}_{A_n}\otimes x_n\right\|_{L^p(S;X)},
\end{aligned}$$

for $q < \infty$ (or the same with the L^∞-norm for $q = \infty$).

Now, simple functions are dense in $L^p(S; X)$, so $E \otimes I_X$ extends uniquely to a bounded operator from $L^p(S; X)$ to $L^q(T; X)$ with norm $\|E \otimes I_X\|_{L^p(S;X)\to L^q(T;X)} \leq \|E\|_{L^p(S)\to L^q(T)}$. For the other direction of the inequality, consider functions of the form $g \otimes x$ with $g \in L^p(S)$ and $x \in X$ with $\|x\|_X = 1$. □

The scalar conditional expectation is a contraction on $L^1(S)$. By monotonicity, $f \geq 0$ implies $\mathbb{E}[f \mid \mathcal{F}] \geq 0$, so we can apply the previous lemma, which gives us that $\mathbb{E}[\cdot \mid \mathcal{F}] \otimes I_X$ extends uniquely to a contractive operator E on $L^1(S; X)$. It remains to show that E satisfies Definition 2.1.2.

For simple functions, $E(g \otimes x) = \mathbb{E}[g \mid \mathcal{F}] \otimes x$. For other functions, we can use the same limiting argument as the scalar case to show that eq. (2.1.1) holds. Therefore, the extension is indeed the conditional expectation on $L^1(S; X)$. This justifies the slight abuse of notation in writing $\mathbb{E}[\cdot \mid \mathcal{F}]$ for the Banach space conditional expectation as well as the scalar conditional expectation. Now, for every $f \in L^1(S; X)$, we have that $\mathbb{E}[f \mid \mathcal{F}] \in L^1(S, \mathcal{F}; X)$ is the unique conditional expectation of f with respect to $\mathcal{F}$. Moreover, it satisfies

$$\|\mathbb{E}[f \mid \mathcal{F}]\|_{L^1(S;X)} \leq \|f\|_{L^1(S;X)}$$

and

$$\int_F \mathbb{E}[f \mid \mathcal{F}] \, \mathrm{d}\mu = \int_F f \, \mathrm{d}\mu$$

for all $F \in \mathcal{F}$. Next, we wish to extend this operator from $f \in L^1(S; X)$ to σ-integrable $f \in L^0(S; X)$.

Theorem 2.1.8 (Existence of conditional expectation)**.** *Suppose that $f \in L^0(S; X)$ is σ-integrable over a σ-finite sub-σ-algebra $\mathcal{F} \subseteq \mathcal{A}$. Then, there exists a unique conditional expectation in $L^0(S, \mathcal{F}; X)$ of f with respect to $\mathcal{F}$. Denoting it by $\mathbb{E}[f \mid \mathcal{F}]$, we have that $\mathcal{F}_f \subseteq \mathcal{F}_{\mathbb{E}[f|\mathcal{F}]}$ and*

$$\int_F \mathbb{E}[f \mid \mathcal{F}] \, \mathrm{d}\mu = \int_F f \, \mathrm{d}\mu$$

for any $F \in \mathcal{F}_f$.

Proof. Fix a pairwise disjoint exhausting sequence $(F_n)_{n\in\mathbb{N}}$ for f in $\mathcal{F}$. As $\mathbb{1}_{F_n} f \in L^1(S; X)$ for all $n \in \mathbb{N}$, we have by the discussion above this theorem that each has a conditional expectation, so

$$g := \sum_{n=1}^{\infty} \mathbb{1}_{F_n} \mathbb{E}[\mathbb{1}_{E_n} f \mid \mathcal{F}]$$

is a well-defined function. By Lemma 2.1.3, $\mathcal{C}_n = \mathcal{F}_{\mathbb{1}_{F_n} f} \cap \mathcal{F}_{\mathbb{E}[\mathbb{1}_{F_n} f|\mathcal{F}]}$ is an exhausting ideal in $\mathcal{F}$ for each $n \in \mathbb{N}$. Then,

$$\mathcal{C} = \bigcup_{n\in\mathbb{N}} \{C \cap F_n \mid C \in \mathcal{C}_n\}$$

is also an exhausting ideal in $\mathcal{F}$, so Lemma 2.1.3 implies that g is in fact a conditional expectation of f with respect to $\mathcal{F}$. Uniqueness follows from Theorem 2.1.6, so we can write $\mathbb{E}[f \mid \mathcal{F}] = g$.

Let $F \in \mathcal{F}_f$, so $\mathbb{1}_F f \in L^1(S; X)$ and has a conditional expectation with respect to $\mathcal{F}$. Then, for any $G \in \mathcal{F}_f \cap \mathcal{F}_g$,

$$\int_G \mathbb{1}_F f \,\mathrm{d}\mu = \int_{F\cap G} f \,\mathrm{d}\mu = \int_{F\cap G} \mathbb{E}[f \mid \mathcal{F}] \,\mathrm{d}\mu = \int_G \mathbb{1}_F \mathbb{E}[f \mid \mathcal{F}] \,\mathrm{d}\mu,$$

so $\mathbb{1}_F \mathbb{E}[f \mid \mathcal{F}]$ is the conditional expectation of $\mathbb{1}_F f$ with respect to $\mathcal{F}$. Then, $\mathbb{1}_F \mathbb{E}[f \mid \mathcal{F}] \in L^1(S; X)$ and satisfies

$$\int_F \mathbb{E}[f \mid \mathcal{F}] \,\mathrm{d}\mu = \int_S \mathbb{1}_F \mathbb{E}[f \mid \mathcal{F}] \,\mathrm{d}\mu = \int_S \mathbb{1}_F f \,\mathrm{d}\mu = \int_F f \,\mathrm{d}\mu,$$

which proves the claim. □

With this theorem, we have successfully defined conditional expectations with respect to σ-finite sub-σ-algebras for Banach space-valued functions which are σ-integrable over the sub-σ-algebra.

By the same arguments as the scalar case, the Banach space conditional expectation inherits most of the useful properties that we have come to expect: dominated convergence, Jensen's inequality, L^p-contractivity, and the tower property. For the details of these theorems, see [HvVW16, Section 2.6]. Our next order of business is to use our construction of conditional expectations to define the notion of a Banach space-valued martingale.

2.2 Martingales and their analysis

Having constructed conditional expectations, we can now turn to the stochastic basis for the UMD and randomized UMD properties. There are two probabilistic ingredients in the definitions of the properties: martingales and Rademacher sequences. For the remainder of this chapter, we study the behavior of each of these two objects. The first section covers martingales and Doob's maximal inequalities, while the second covers Rademacher sequences, Kahane's contraction principle, and the Kahane-Khintchine inequalities.

2.2.1 Martingales

Equipped with conditional expectation operators, we proceed to define Banach space-valued martingales in much the same way as $\mathbb{R}$-valued martingales. First, we define a *filtration* of σ-algebras on a measurable space and sequences of random variables which are *adapted* to a filtration.

Definition 2.2.1 (Filtration)**.** *A* filtration *on a measurable space* $(S, \mathcal{A})$ *is a sequence* $(\mathcal{F})_{n=1}^N$ *of sub-*σ*-algebras of* $\mathcal{A}$ *such that* $\mathcal{F}_m \leq \mathcal{F}_n$ *for all indices* $m \leq n$*. A sequence of functions* $(f_n)_{n=1}^N$ *in* $L^0(S; X)$ *is called* adapted *to* $(\mathcal{F}_n)_{n=1}^N$ *if* f_n *is* $\mathcal{F}_n$*-measurable for all* $n = 1, \ldots, N$*.*

These definitions of filtrations and adapted sequences give rise to a natural extension of the definition of martingales to the Banach space setting.

Definition 2.2.2 (Martingale)**.** *Let* $(S, \mathcal{A}, \mu)$ *be a* σ*-finite measure space with* σ*-finite filtration* $(\mathcal{F}_n)_{n=1}^N$ *and* X *be a Banach space. A sequence of functions* $(f_n)_{n=1}^N$ *in* $L^0(S; X)$ *is called a* martingale *if it is adapted to* $(\mathcal{F}_n)_{n=1}^N$ *and, for all indices* $m \leq n$*,* f_n *is* σ*-integrable over* $\mathcal{F}_m$ *and satisfies* $\mathbb{E}\left[f_n \mid \mathcal{F}_m\right] = f_m$*. If the sequence lies in* $L^p(S; X)$ *for some* $p \in [1, \infty]$*, it is called an* L^p-martingale*.*

We can slightly relax this definition in two ways. We call a sequence of functions $(f_n)_{n=1}^N$ which satisfies all other properties of a martingale but only has $\mathbb{E}\left[f_n \mid \mathcal{F}_m\right] \geq f_m$ a *submartingale*. Similarly, a sequence $(f_n)_{n=1}^N$ which satisfies all other properties but only has $\mathbb{E}\left[f_n \mid \mathcal{F}_m\right] \leq f_m$ is called a *supermartingale*. Of course, a sequence is a martingale if and only if it is both a submartingale and a supermartingale.

UMD spaces and their randomized counterparts are defined by the behavior of *difference sequences* of martingales, which are given by the following definition.

Definition 2.2.3 (Martingale difference sequence)**.** *Let* $(f_n)_{n=1}^N$ *be an* X*-valued mar-*

tingale. The sequence $(df_n)_{n=1}^N$ *defined by* $df_n := f_n - f_{n-1}$ *(with the convention that* $f_0 = 0$*) is called the* martingale difference sequence *associated with* $(f_n)_{n=1}^N$*. If* $(f_n)_{n=1}^N$ *is an* L^p*-martingale,* $(df_n)_{n=1}^N$ *is called an* L^p-martingale difference sequence*.*

Our definitions give way to another simple characterization of martingale difference sequences without the martingales which generate them. Given an arbitrary sequence of functions $(d_n)_{n=1}^N$ in $L^0(S;X)$, it is a martingale difference sequence (for some martingale on a σ-finite measure space $(S,\mathcal{A},\mu)$ with filtration $(\mathcal{F}_n)_{n=1}^N$) if and only if it is adapted, σ-integrable with respect to $(\mathcal{F}_n)_{n=1}^N$, and $\mathbb{E}\left[d_n \mid \mathcal{F}_{n-1}\right] = 0$ for all $n = 1, \dots, N$ (with the convention that $\mathcal{F}_0$ is the trivial σ-algebra).

It is easy to check that, when H is a Hilbert space, $L^2(S;H)$-martingale difference sequences are orthogonal, which we do in the following proposition.

Proposition 2.2.4. *Let* H *be a Hilbert space and* $(S,\mathcal{A},\mu)$ *be a* σ*-finite measure space. Then, any* $L^2(S;H)$*-martingale difference sequence* $(df_n)_{n=1}^N$ *is mutually orthogonal. In particular,*

$$\left\| \sum_{n=1}^N df_n \right\|_{L^2(S;H)} = \left(\sum_{n=1}^N \|df_n\|_{L^2(S;H)}^2 \right)^{\frac{1}{2}}.$$

Proof. For $m < n$, $\mathbb{E}[df_n \mid \mathcal{F}_m] = 0$ and df_m is $\mathcal{F}_m$-measurable, so

$$\int_S \langle df_n, df_m \rangle_H \,\mathrm{d}\mu = \int_S \mathbb{E}[\langle df_n, df_m \rangle_H \mid \mathcal{F}_m] \,\mathrm{d}\mu = \int_S \langle \mathbb{E}[df_n \mid \mathcal{F}_m], df_m \rangle_H = 0.$$

That is, $(df_n)_{n=1}^N$ is mutually orthogonal. Then, we have by the Pythagorean theorem that

$$\left\| \sum_{n=1}^N df_n \right\|_{L^2(S;H)} = \left(\sum_{n=1}^N \|df_n\|_{L^2(S;H)}^2 \right)^{\frac{1}{2}},$$

as desired. □

One of the most convenient characteristics of Hilbert spaces is the existence of orthogonal sequences. The proposition which we have just proven implies that it may be possible to adopt this notion in the more general Banach space setting by substituting martingale difference sequences for orthogonal sequences. In the setting of the proposition, we have the identity

$$\left\| \sum_{n=1}^N \epsilon_n df_n \right\|_{L^2(S;H)} = \left\| \sum_{n=1}^N df_n \right\|_{L^2(S;H)}$$

for scalars $|\epsilon_n| = 1$, $n = 1, \ldots, N$. Up to some constant, the UMD property given by Definition 1.1.1 extends this notion from $L^2(S;H)$ to $L^p(S;X)$, where X is not necessarily a Hilbert space.

For the rest of this section, we study the analysis of martingales: their approximation and estimates for their norms.

2.2.2 Approximation of martingales

In the previous section, we gave a slightly more general notion of martingales than usual. Rather than define martingales on probability spaces (i.e. measure spaces for which $\mu(S) = 1$), we chose to define them on the more general class of σ-finite measure spaces. The reason for this choice was that one may encounter σ-finite measure spaces which are not probability spaces in harmonic analysis.

That being said, the following lemma shows that it is possible to approximate martingales on σ-finite measure spaces by simple martingales whose supports have finite measure. The proof is based on that of [Pis16, Lemma 5.41].

Lemma 2.2.5. *Let $(f_n)_{n=1}^N$ be an $L^p(S;X)$-martingale for a σ-finite measure space $(S, \mathcal{A}, \mu)$ adapted to a σ-finite filtration $(\mathcal{F}_n)_{n=1}^N$. For any $\epsilon > 0$, there exists a sequence of functions $(g_n)_{n=1}^N$ from S to X such that:*

(1) for $n = 1, \ldots, N$, g_n is a simple function which is supported on a set $E \in \mathcal{F}_1$ with $\mu(E) < \infty$,

(2) the sequence formed by restricting g_n to E for $n = 1, \ldots, N$ forms a martingale with respect to a filtration of finite σ-algebras on E,

(3) $\|f_n - g_n\|_{L^p(S;X)} < \epsilon$ for $n = 1, \ldots, N$.

Proof. As S is σ-finite, the dominated convergence theorem gives us an $E \in \mathcal{F}_1$ with $\mu(E) < \infty$ such that $\|\mathbb{1}_{E^c} f_n\|_{L^p(S;X)} < \delta$ for any $\delta > 0$. Then, for $n, m = 1, \ldots, N$,

$$\mathbb{E}[\mathbb{1}_E f_n \mid \mathcal{F}_m] = \mathbb{1}_E \mathbb{E}[f_n \mid \mathcal{F}_m],$$

so $(\mathbb{1}_E f_n)_{n=1}^N$ is also a martingale. Replacing $(f_n)_{n=1}^N$ with $(\mathbb{1}_E f_n)_{n=1}^N$, we can suppose without loss of generality that f_n is supported on the finite measure set E.

Now, let $\delta > 0$. By density of simple functions in $L^p(S;X)$, there exist $s_n \in L^p(S;X)$ with $\|s_n - df_n\|_{L^p(S;X)} < \delta$ for $n = 1, \ldots, N$.

Consider the filtration of finite σ-algebras $(\mathcal{G}_n)_{n=1}^N$ given by

$$\mathcal{G}_1 := \sigma(E, s_1) \subseteq \mathcal{F}_1, \quad \mathcal{G}_n := \sigma(\mathcal{G}_{n-1}, s_n) \subseteq \mathcal{F}_n,$$

so that $(s_n)_{n=1}^N$ is adapted to $(\mathcal{G}_n)_{n=1}^N$ on E.

Define the martingale $(g_n)_{n=1}^N$ by its difference sequence $dg_n = s_n - \mathbb{E}[s_n \mid \mathcal{G}_{n-1}]$, which is thus adapted to $(\mathcal{G}_n)_{n=1}^N$ on E.

Now, for $n = 1, \dots, N$,

$$\mathbb{E}[df_n \mid \mathcal{G}_{n-1}] = \mathbb{E}[\mathbb{E}[df_n \mid \mathcal{F}_{n-1}] \mid \mathcal{G}_{n-1}] = \mathbb{E}[0 \mid \mathcal{G}_{n-1}] = 0$$

and

$$\|dg_n - df_n\|_{L^p(S;X)} = \|s_n - df_n - \mathbb{E}[s_n - df_n \mid \mathcal{G}_{n-1}]\|_{L^p(S;X)} \leq 2\|s_n - df_n\|_{L^p(S;X)} < 2\delta,$$

so, summing the difference sequences, $\|g_n - f_n\|_{L^p(S;X)} < (1+2n)\delta \leq (1+2N)\delta$. Our choice of $\delta > 0$ was arbitrary, so this proves the claim. $\square$

Later on, we will use this fact to show that the UMD$^+$ and UMD$^-$ properties follow from considering the behavior of simple martingales whose images are contained in finite-dimensional subspaces of X. In the next subsection, our study of martingales concludes with proofs for Doob's maximal inequalities.

2.2.3 Doob's maximal inequalities

Martingales are defined by the fundamental equality $\mathbb{E}\left[f_n \mid \mathcal{F}_m\right] = f_m$, but from this equality arise several important inequalities which are useful for their analysis. First among these are Doob's classical maximal inequalities first stated in [Doo90], which control the *maximal functions* of martingales and scalar submartingales: the maxima of their norms. In this subsection, we prove Doob's inequalities first in the scalar case, then for Banach space-valued martingales.

We begin with the definition of the maximal function of a sequence of functions (in particular, of a martingale).

Definition 2.2.6 (Maximal function)**.** *For a sequence of functions $(f_n)_{n=1}^N$ from S to a Banach space X and some fixed $1 \leq M \leq N$, the* maximal function $f_M^* : S \to [0, \infty)$ *is given by*

$$f_M^* := \max_{1 \leq n \leq M} \|f_n\|_X.$$

Before proceeding to Doob's maximal inequalities in the Banach space-valued setting, we mention the inequality for the scalar submartingale case.

Lemma 2.2.7. *Let* $(f_n)_{n=1}^N$ *be a non-negative scalar submartingale. Then, for all* $1 \leq M \leq N$ *and any* $\lambda > 0$*,*

$$\mu(f_M^* > \lambda) \leq \frac{1}{\lambda} \int_{\{f_M^* > \lambda\}} f_M \, \mathrm{d}\mu.$$

Proof. Define the stopping time $\tau : S \to \{1, \dots, M+1\}$ by

$$\tau := \min\{n \in \{1, \dots, M\} \mid f_n > \lambda\}$$

with the convention that $\min \emptyset = M + 1$. Then, $\{f_M^* > \lambda\} = \{\tau \leq M\}$. We can compute

$$\begin{aligned}
\lambda\mu(f_M^* > \lambda) &= \lambda \int_{\{\tau \leq M\}} \mathrm{d}\mu \\
&= \lambda \sum_{n=1}^{M} \int_{\{\tau = n\}} \mathrm{d}\mu \\
&\leq \sum_{n=1}^{M} \int_{\{\tau = n\}} f_n \, \mathrm{d}\mu \\
&\leq \sum_{n=1}^{M} \int_{\{\tau = n\}} \mathbb{E}\left[f_M \mid \mathcal{F}_n\right] \mathrm{d}\mu \\
&= \sum_{n=1}^{M} \int_{\{\tau = n\}} f_M \, \mathrm{d}\mu \\
&= \int_{\{f_M^* > \lambda\}} f_M \, \mathrm{d}\mu,
\end{aligned}$$

using that $(f_n)_{n=1}^N$ is a submartingale and the definition of conditional expectation. □

Now, we proceed to reduce Doob's maximal inequalities to a statement about scalar submartingales, then apply the lemma above to complete the proof. Here, we consider the $p = 1$ and $p > 1$ cases separately; the latter follows directly from the former. Qualitatively, their content is that the maximum of an L^p-martingale is also in L^p.

Theorem 2.2.8 (Doob's maximal inequalities)**.** *Let X be a Banach space, fix $p \in [1, \infty]$, and let $(f_n)_{n=1}^N$ be a non-negative scalar $L^p(S)$-submartingale or an $L^p(S;X)$-martingale for a σ-finite measure space $(S, \mathcal{A}, \mu)$. Then, for any $\lambda > 0$ and $1 \leq M \leq N$,*

$$\mu(f_M^* > \lambda) \leq \frac{1}{\lambda}\|f_M\|_{L^1(S;X)}, \quad \text{if } p = 1,$$
$$\|f_M^*\|_{L^p(S)} \leq p'\|f_M\|_{L^p(S;X)}, \quad \text{if } p \in (1, \infty].$$

Proof. If $(f_n)_{n=1}^N$ is an $L^p(S;X)$-martingale, $(F_n)_{n=1}^N$ given by $F_n = \|f_n\|_X$ is a non-negative scalar L^p-submartingale, so it suffices to consider only the case where $(f_n)_{n=1}^N$ is a non-negative scalar L^p-submartingale. Using the previous lemma,

$$\mu(f_M^* > \lambda) \leq \frac{1}{\lambda}\int_{\{f_M^* > \lambda\}} f_M \,\mathrm{d}\mu \leq \frac{1}{\lambda}\|f_M\|_{L^1(S)},$$

which is the first inequality.

Applying the previous equation again,

$$\begin{aligned}
\|f_M^*\|_{L^p(S)}^p &= \int_0^\infty p\lambda^{p-1}\mu(f_M^* > \lambda)\,\mathrm{d}\lambda \\
&\leq \int_0^\infty p\lambda^{p-2}\int_{\{f_M^* > \lambda\}} f_M \,\mathrm{d}\mu\,\mathrm{d}\lambda \\
&= \int_S \left(\int_0^{f_M^*(s)} p\lambda^{p-2}\,\mathrm{d}\lambda\right) f_M(s)\,\mathrm{d}\mu(s) \\
&= p'\int_S f_M(f_M^*)^{p-1}\,\mathrm{d}\mu \\
&\leq p'\|f_M\|_{L^p(S)}\|f_M^*\|_{L^p(S)}^{p-1}.
\end{aligned}$$

Dividing by $\|f_M^*\|_{L^p(S)}^{p-1}$ on both sides (which is finite, as $(f_n)_{n=1}^N$ is an L^p-submartingale), we obtain

$$\|f_M^*\|_{L^p(S)} \leq p'\|f_M\|_{L^p(S)},$$

which is the second inequality. □

In the general theory of martingales, Doob's maximal inequalities are immensely useful for proving results about the convergence of L^p-martingales. See [Cha64, IN68] for examples of such usage. They also have applications to harmonic analysis via the Fefferman-Stein inequality of [FS71]. We will later use Doob's maximal inequalities for the computation of randomized UMD constants.

2.3 Rademacher sequences

As we have seen, the UMD and randomized UMD properties are defined using the norms of Rademacher sums of martingale difference sequences. We treated martingales in the previous section; in this section, we study Rademacher sequences. We begin with the definitions.

Definition 2.3.1 (Rademacher random variable and sequence)**.** *A* Rademacher random variable *is a random variable $\varepsilon : \Omega \to K$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is uniformly distributed over $S_{\mathbb{K}} := \{z \in \mathbb{K} \mid |z| = 1\}$ (where $\mathbb{K}$ is the underlying field). A* Rademacher sequence *is a sequence $(\varepsilon_n)_{n=1}^N$ of independent Rademacher random variables.*

We will typically denote Rademacher random variables taking values in $S_{\mathbb{K}}$ by ε and real Rademacher variables (i.e. those taking values in $\{-1, 1\}$ even when the scalar field is $\mathbb{C}$) by r. Complex Rademacher random variables are typically known in the literature as *Steinhaus* random variables, but we will not use this term.

The following subsections detail some inequalities involving Rademacher sequences which we will use during our examination of UMD spaces and their properties in the sequel.

2.3.1 Kahane's contraction principle

We begin by proving Kahane's contraction principle, which enables the factoring out of scalar coefficients from Rademacher sums of elements in a Banach space. The original inequality was obtained in [Kah85], while the variant for real Rademacher sequences is from [PW98, Principle 3.5.4]. Both follow by convexity arguments, for which we begin with the definitions of convex and absolute convex hulls. Our proof is modeled off of that of [HvVW16, Proposition 3.2.10].

Definition 2.3.2 (Convex, absolute convex hulls)**.** *Let T be a subset of a vector space V.*

(1) The convex hull *of T,* $\mathrm{conv}(T)$*, is the set of all vectors of the form $\sum_{j=1}^k \lambda_j x_j$, where $\lambda_1, \ldots, \lambda_k \in [0, 1]$ sum to one and $x_1, \ldots, x_k \in T$.*

(2) The absolute convex hull *of T with respect to scalar field $\mathbb{K}$,* $\mathrm{conv}_{\mathbb{K}}(T)$*, is the set of all vectors of the form $\sum_{j=1}^k \lambda_j x_j$, where $\lambda_1, \ldots, \lambda_k \in \mathbb{K}$ with $\sum_{j=1}^k |\lambda_j| \le 1$ and $x_1, \ldots, x_k \in T$.*

It is easy to check that conv commutes with Cartesian products, which we do in the following lemma. Note that the same statement is not necessarily true if conv is replaced by abco.

Lemma 2.3.3. *Let $X_1, \dots, X_N$ be vector spaces and let $E_n \subseteq V_n$ for $n = 1, \dots, N$. Then,*

$$\operatorname{conv}(E_1 \times \cdots \times E_N) = \operatorname{conv}(E_1) \times \cdots \times \operatorname{conv}(E_N).$$

Proof. By iteratively taking Cartesian products, we can suppose without loss of generality that $N = 2$. Clearly, the left-hand side of the equality is a subset of the right-hand side. For the other direction, let $(x, y) \in \operatorname{conv}(E_1) \times \operatorname{conv}(E_2)$. Then, we can write

$$x = \sum_{i=1}^{k} \lambda_i x_i, \quad y = \sum_{j=1}^{l} \mu_j y_j$$

for $x_1, \dots, x_k \in E_1$, $y_1, \dots, y_l \in E_2$, and $\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_l \in [0, 1]$ with $\sum_{i=1}^{k} \lambda_i = \sum_{j=1}^{l} \mu_j = 1$. This implies that $\sum_{i=1}^{k} \sum_{j=1}^{l} \lambda_i \mu_j = 1$ as well, so

$$(x, y) = \left(\sum_{i=1}^{k} \lambda_i x_i, \sum_{j=1}^{l} \mu_j y_j \right) = \sum_{i=1}^{k} \sum_{j=1}^{l} \lambda_i \mu_j (x_i, y_j) \in \operatorname{conv}(E_1 \times E_2),$$

which proves the claim. □

We now take a brief detour to mention the following consequence of the Hahn-Banach theorem, which we use to relate $(\overline{B}_{\mathbb{C}})^N$ with $\operatorname{abco}_{\mathbb{C}}(\{-1, 1\}^N)$ for the proof of the real Rademacher sequence variant from [PW98].

Lemma 2.3.4. *Let X and Y be Banach spaces and let $T : X \to Y$ be a bounded linear operator such that there exists a $\delta > 0$ such that $\|Tx\|_Y \geq \delta \|x\|_X$ for all $x \in X$. Then,*

$$\delta \overline{B}_{X^*} \subseteq T^* \overline{B}_{Y^*}.$$

Proof. Suppose $\dim(X) \geq 1$, otherwise the claim is trivial. Let $Y_0 := TX$, which is a closed subspace of Y. Fix $x_0^* \in X^*$ with $\|x_0^*\|_{X^*} = 1$ and define $y_0^* : Y_0 \to \mathbb{K}$ by $\langle Tx, y_0^* \rangle := \langle x, x_0^* \rangle$. Then,

$$|\langle Tx, y_0^* \rangle| \leq \|x\|_X \|x_0^*\|_{X^*} \leq \frac{1}{\delta} \|Tx\|_Y,$$

so $y_0^* \in Y_0^*$ and $\|y_0^*\|_{Y^*} \leq \frac{1}{\delta}$. By the Hahn-Banach theorem, y_0^* extends to $y^* \in Y^*$ with $\|y^*\|_{Y^*} \leq \frac{1}{\delta}$. Then,

$$\langle x, T^* y^* \rangle = \langle Tx, y^* \rangle = \langle Tx, y_0^* \rangle = \langle x, x_0^* \rangle$$

for all $x \in X$, so $\delta x_0^* = T^*(\delta y^*) \in T^* \overline{B}_{Y^*}$, which proves the claim. □

Applying the lemma above with a particular choice of operator T, the following lemma shows that $(\overline{B}_{\mathbb{C}})^N$ is a subset of $\operatorname{abco}_{\mathbb{C}}(\{-1,1\}^N)$, after scaling.

Lemma 2.3.5. *We have that*

$$(\overline{B}_{\mathbb{R}})^N = \operatorname{conv}(S_{\mathbb{R}}^N) = \operatorname*{abco}_{\mathbb{R}}(\{-1,1\}^N),$$
$$(\overline{B}_{\mathbb{C}})^N = \operatorname{conv}(S_{\mathbb{C}}^N) \subseteq \frac{\pi}{2} \operatorname*{abco}_{\mathbb{C}}(\{-1,1\}^N).$$

Proof. The two equalities on the left follow from $\overline{B}_{\mathbb{K}} = \operatorname{conv}(S_{\mathbb{K}})$ and Lemma 2.3.3, while the equality in the upper right follows from the fact that $\operatorname{conv}(S_{\mathbb{R}}^N)$ is absolutely convex.

For the inclusion, define the operator $T : \ell_N^1 \to \ell^\infty(\{-1,1\}^N)$ by

$$(Tx)(\epsilon) = \sum_{n=1}^{N} \epsilon_n x_n$$

which has adjoint $T^* : \ell^1(\{-1,1\}^N) \to \ell_N^\infty$ given by

$$(T^*\lambda)_n = \sum_{\epsilon \in \{-1,1\}^N} \lambda(\epsilon) \epsilon_n.$$

Consider the polar decomposition $x_n = |x_n| e^{2\pi i t_n}$. Then,

$$\begin{aligned}
\|Tx\|_{\ell^\infty(\{-1,1\}^N)} &= \sup_{\epsilon \in \{-1,1\}^N} \sup_{t \in [0,1)} \left| \mathrm{Re}\left(e^{2\pi i t} \sum_{n=1}^N \epsilon_n x_n \right) \right| \\
&= \sup_{\epsilon \in \{-1,1\}^N} \sup_{t \in [0,1)} \left| \sum_{n=1}^N \epsilon_n |x_n| \cos(2\pi(t_n + t)) \right| \\
&= \sup_{t \in [0,1)} \sum_{n=1}^N |x_n| |\cos(2\pi(t_n + t))| \\
&\geq \int_0^1 \sum_{n=1}^N |x_n| |\cos(2\pi(t_n + t))| \, \mathrm{d}t \\
&= \frac{2}{\pi} \sum_{n=1}^N |x_n| \\
&= \frac{2}{\pi} \|x\|_{\ell^1_N}.
\end{aligned}$$

By the previous lemma, this implies that $\overline{B}_{\ell^\infty_N} \subseteq \frac{\pi}{2} T^* \overline{B}_{\ell^1(\{-1,1\}^N)}$, but the former is $\mathrm{conv}(S_{\mathbb{C}}^N)$ and the latter is $\frac{\pi}{2} \, \mathrm{abco}_{\mathbb{C}}(\{-1,1\}^N)$. □

Now, we use the characterizations of $(\overline{B}_{\mathbb{K}})^N$ in the previous lemma to prove the following deterministic variant of Kahane's contraction principle, which takes the supremum over all $\epsilon \in (S_{\mathbb{K}})^N$ or $\{-1,1\}^N$. For the contraction principle, we will replace this supremum with an expectation.

Lemma 2.3.6. *For all sequences of scalars $(a_n)_{n=1}^N$ and finite sequences $(x_n)_{n=1}^N$ in X,*

$$\left\| \sum_{n=1}^N a_n x_n \right\|_X \leq \max_{1 \leq n \leq N} |a_n| \sup_{\epsilon \in (S_{\mathbb{K}})^N} \left\| \sum_{n=1}^N \epsilon_n x_n \right\|_X$$

and

$$\left\| \sum_{n=1}^N a_n x_n \right\|_X \leq \frac{\pi}{2} \max_{1 \leq n \leq N} |a_n| \sup_{\epsilon \in \{-1,1\}^N} \left\| \sum_{n=1}^N \epsilon_n x_n \right\|_X.$$

Proof. We begin with the first part. By scaling and using homogeneity of the norm, we can suppose without loss of generality that $\max_{1 \leq n \leq N} |a_n| = 1$. Then, for $n = 1, \ldots, N$, $a_n \in \mathrm{conv}(S_{\mathbb{K}})$, so $(a_n)_{n=1}^N \in \mathrm{conv}(S_{\mathbb{K}}^N)$. This means that there exist

$\lambda_1, \ldots, \lambda_k \in [0,1]$ summing to at most one and $(\epsilon_n^{(1)})_{n=1}^N, \ldots, (\epsilon_n^{(k)})_{n=1}^N \in S_{\mathbb{K}}^N$ such that

$$(a_n)_{n=1}^N = \sum_{j=1}^k \lambda_j (\epsilon_n^{(j)})_{n=1}^N.$$

Now, we can compute

$$\begin{aligned}
\left\| \sum_{n=1}^N a_n x_n \right\|_X &= \left\| \sum_{n=1}^N \sum_{j=1}^k \lambda_j \epsilon_n^{(j)} x_n \right\|_X \\
&\leq \sum_{j=1}^k \lambda_j \left\| \sum_{n=1}^N \epsilon_n^{(j)} x_n \right\|_X \\
&\leq \sum_{j=1}^k \lambda_j \sup_{\epsilon \in (S_{\mathbb{K}})^N} \left\| \sum_{n=1}^N \epsilon_n x_n \right\|_X \\
&\leq \sup_{\epsilon \in (S_{\mathbb{K}})^N} \left\| \sum_{n=1}^N \epsilon_n x_n \right\|_X,
\end{aligned}$$

which proves the first part.

Next, we prove the second part. Scaling once again, we can suppose without loss of generality that $\max_{1 \leq n \leq N} |a_n| = 1$. Then, using the same argument as the first part as well as the inclusion statement from the previous lemma, $(a_n)_{n=1}^N \in \frac{\pi}{2} \operatorname{abco}_{\mathbb{C}}(\{-1,1\}^N)$. This means that there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ with $\sum_{j=1}^k |\lambda_j| \leq \frac{\pi}{2}$ and $(\epsilon_n^{(1)})_{n=1}^N, \ldots, (\epsilon_n^{(k)})_{n=1}^N \in \{-1,1\}^N$ such that

$$(a_n)_{n=1}^N = \sum_{j=1}^k \lambda_j (\epsilon_n^{(j)})_{n=1}^N.$$

As before, we can compute

$$\begin{aligned}\left\|\sum_{n=1}^{N} a_n x_n\right\|_X &= \left\|\sum_{n=1}^{N}\sum_{j=1}^{k} \lambda_j \epsilon_n^{(j)} x_n\right\|_X \\ &\leq \sum_{j=1}^{k} |\lambda_j| \left\|\sum_{n=1}^{N} \epsilon_n^{(j)} x_n\right\|_X \\ &\leq \sum_{j=1}^{k} |\lambda_j| \sup_{\epsilon\in\{-1,1\}^N} \left\|\sum_{n=1}^{N} \epsilon_n x_n\right\|_X \\ &\leq \frac{\pi}{2} \sup_{\epsilon\in\{-1,1\}^N} \left\|\sum_{n=1}^{N} \epsilon_n x_n\right\|_X,\end{aligned}$$

which proves the second part. □

Armed with this lemma, Kahane's contraction principle follows rather easily.

Theorem 2.3.7 (Kahane's contraction principle)**.** *For all sequences of scalars $(a_n)_{n=1}^N$, finite sequences $(x_n)_{n=1}^N$ in X, and $p \in [1,\infty]$,*

$$\left\|\sum_{n=1}^{N} a_n \varepsilon_n x_n\right\|_{L^p(\Omega;X)} \leq \max_{1\leq n\leq N} |a_n| \left\|\sum_{n=1}^{N} \varepsilon_n x_n\right\|_{L^p(\Omega;X)},$$

where $(\varepsilon_n)_{n=1}^N$ is a Rademacher sequence on a probability space Ω.

If X is a complex Banach space, then we also have that

$$\left\|\sum_{n=1}^{N} a_n r_n x_n\right\|_{L^p(\Omega;X)} \leq \frac{\pi}{2} \max_{1\leq n\leq N} |a_n| \left\|\sum_{n=1}^{N} r_n x_n\right\|_{L^p(\Omega;X)},$$

where $(r_n)_{n=1}^N$ is a real Rademacher sequence on a probability space Ω.

Proof. For the first part, apply the first part of the previous lemma with $L^p(\Omega;X)$ in place of X and $\varepsilon_n x_n$ in place of x_n, then notice that the sequences $(\epsilon_n \varepsilon_n)_{n=1}^N$ and $(\varepsilon_n)_{n=1}^N$ are identically distributed for any choice of $(\epsilon_n)_{n=1}^N$ with $|\epsilon_n| = 1$ for $n = 1, \ldots, N$.

For the second part, apply the second part of the previous lemma with $L^p(\Omega;X)$ in place of X and $r_n x_n$ in place of x_n, then notice that the sequences $(\epsilon_n r_n)_{n=1}^N$ and $(r_n)_{n=1}^N$ are identically distributed for any choice of $(\epsilon_n)_{n=1}^N \in \{-1,1\}^N$. □

The second part of Kahane's contraction principle gives rise to the following convenient comparison between real and complex Rademacher sums, which we will use to prove the Kahane-Khintchine inequalities in the next subsection and later to study the impact of the underlying scalar field on the randomized UMD properties and K-convexity.

Corollary 2.3.8. *Let X be a complex Banach space and let $(r_n)_{n=1}^N$ and $(\varepsilon_n)_{n=1}^N$ be real and complex Rademacher sequences, respectively, on a probability space Ω. Then, for all finite sequences $(x_n)_{n=1}^N$ in X and $p \in (1, \infty)$,*

$$\frac{2}{\pi}\left\|\sum_{n=1}^N r_n x_n\right\|_{L^p(\Omega;X)} \leq \left\|\sum_{n=1}^N \varepsilon_n x_n\right\|_{L^p(\Omega;X)} \leq \frac{\pi}{2}\left\|\sum_{n=1}^N r_n x_n\right\|_{L^p(\Omega;X)}.$$

Proof. As they do not appear in the same $L^p(\Omega; X)$-norm, we can assume that $(r_n)_{n=1}^N$ and $(\varepsilon_n)_{n=1}^N$ are defined on distinct probability spaces, Ω and $\widetilde{\Omega}$. For any $\widetilde{\omega} \in \widetilde{\Omega}$,

$$\frac{2}{\pi}\left\|\sum_{n=1}^N r_n x_n\right\|_{L^p(\Omega;X)} \leq \left\|\sum_{n=1}^N r_n \varepsilon_n(\widetilde{\omega}) x_n\right\|_{L^p(\Omega;X)} \leq \frac{\pi}{2}\left\|\sum_{n=1}^N r_n x_n\right\|_{L^p(\Omega;X)},$$

using the second part of Kahane's contraction principle with scalars $\overline{\varepsilon}_n(\widetilde{\omega})$ and elements $\varepsilon_n(\widetilde{\omega})x_n$, then scalars $\varepsilon_n(\widetilde{\omega})$ and elements x_n. Taking $L^p(\widetilde{\Omega})$-norms and using that the sequences $(r_n\varepsilon_n)_{n=1}^N$ and $(\varepsilon_n)_{n=1}^N$ are identically distributed, we have the desired inequalities. □

This completes our study of Kahane's contraction principle. In the next subsection, we prove the Kahane-Khintchine inequalities.

2.3.2 Kahane-Khintchine inequalities

In this subsection, we prove the Kahane-Khintchine inequalities, whose content is that the L^p and L^q-norms of Rademacher sums are comparable for distinct $p, q \in [1, \infty)$. Khintchine first proved the scalar-valued version in [Khi23], which Kahane extended to the Banach space-valued setting in [Kah85]. The following proof is due to [Gar07].

We begin with a classical tail estimate known as Lévy's inequality.

Lemma 2.3.9 (Lévy's inequality)**.** *Let $(Y_n)_{n=1}^N$ be a finite sequence of independent, real-symmetric*[1] *X-valued random variables and let $S_n = \sum_{k=1}^n Y_k$ for $n = 1, \dots N$. Then, for all $r \geq 0$,*

$$\mathbb{P}\left(\max_{1\leq n\leq N} \|S_n\|_X > r\right) \leq 2\mathbb{P}\left(\|S_N\|_X > r\right).$$

Proof. For $n = 1, \dots, N$, let

$$A_n := \{\|S_1\| \leq r, \dots, \|S_{n-1}\| \leq r, \|S_n\| > r\},$$

and note that the collection $\{A_n\}_{n=1}^N$ is pairwise disjoint. Then, let

$$A := \bigsqcup_{n=1}^N A_n = \left\{\max_{1\leq n\leq N} \|S_n\| > r\right\}.$$

Now, fix $1 \leq n \leq N$. Notice that

$$S_n = \frac{S_N + (2S_n - S_N)}{2},$$

which implies that

$$\{\|S_n\| > r\} \subseteq \{\|S_N\| > r\} \cup \{\|2S_n - S_N\| > r\}.$$

As each Y_n is symmetric, $(Y_1, \dots, Y_N)$ and $(Y_1, \dots, Y_n, -Y_{n+1}, \dots, -Y_N)$ are identically distributed. We have the identities

$$S_N = S_n + Y_{n+1} + \dots + Y_N \quad \text{and} \quad 2S_n - S_N = S_n - Y_{n+1} - \dots - Y_N,$$

which imply that $(Y_1, \dots, Y_n, S_N)$ and $(Y_1, \dots, Y_n, 2S_n - S_N)$ are also identically distributed.

Combining these two facts,

$$\mathbb{P}(A_n) \leq \mathbb{P}(A_n \cap \{\|S_N\| > r\}) + \mathbb{P}(A_n \cap \{\|2S_n - S_N\| > r\}) = 2\mathbb{P}(A_n \cap \{\|S_N\| > r\}).$$

Summing over all n, we obtain

$$\mathbb{P}(A) = \sum_{n=1}^N \mathbb{P}(A_n) \leq 2\sum_{n=1}^N \mathbb{P}(A_n \cap \{\|S_N\| > r\}) = 2\mathbb{P}(\|S_N\| > r),$$

as desired. □

[1] In the sense that Y_n and $-Y_n$ are identically distributed for $n = 1, \dots, N$.

Choosing $Y_n = r_n x_n$, we can adapt this estimate to control the L^p-norm of real Rademacher sums in particular, which will allow us to prove the Kahane-Khintchine inequalities.

Lemma 2.3.10. *For all finite sequences $(x_n)_{n=1}^N$ in X and $r > 0$,*

$$\mathbb{P}\left(\left\|\sum_{n=1}^N r_n x_n\right\|_X > 2r\right) \leq 4\mathbb{P}\left(\left\|\sum_{n=1}^N r_n x_n\right\|_X > r\right)^2,$$

where $(r_n)_{n=1}^N$ is a real Rademacher sequence on a probability space Ω.

Proof. For $n = 1, \ldots, N$, let $S_n = \sum_{k=1}^n r_k x_k$ and, as in the previous lemma,

$$A_n := \{\|S_1\| \leq r, \ldots, \|S_{n-1}\| \leq r, \|S_n\| > r\},.$$

Notice that $(r_1, \ldots, r_N)$ and $(r_1, \ldots, r_n, r_n r_{n+1}, \ldots r_n r_N)$ are identically distributed and that $|r_n| = 1$ almost surely. Therefore, we can write

$$\begin{aligned}
\mathbb{P}(A_n \cap \{\|S_N - S_{n-1}\| > r\}) &= \mathbb{P}\left(A_n \cap \left\{\left\|\sum_{k=n}^N r_k x_k\right\| > r\right\}\right) \\
&= \mathbb{P}\left(A_n \cap \left\{\left\|r_n \sum_{k=n}^N r_k x_k\right\| > r\right\}\right) \\
&= \mathbb{P}\left(A_n \cap \left\{\left\|x_n + \sum_{k=n+1}^N r_n r_k x_k\right\| > r\right\}\right) \\
&= \mathbb{P}\left(A_n \cap \left\{\left\|x_n + \sum_{k=n+1}^N r_k x_k\right\| > r\right\}\right) \\
&= \mathbb{P}\left(A_n \cap \{\|x_n + S_N - S_n\| > r\}\right).
\end{aligned}$$

By the same argument, we also have that

$$\mathbb{P}(\|S_N - S_{n-1}\| > r) = \mathbb{P}(\|x_n + S_N - S_n\| > r).$$

For any $\omega \in A_n$, $\|S_{n-1}(\omega)\| \leq r$, so if $\|S_N(\omega)\| > 2r$, then $\|S_N(\omega) - S_{n-1}(\omega)\| > r$. As $(r_n)_{n=1}^N$ is a sequence of independent random variables, $S_N - S_n$ is independent of

A_n, so

$$\begin{aligned}\mathbb{P}(A_n \cap \{\|S_N\| > 2r\}) &\le \mathbb{P}(A_n \cap \{\|S_N - S_{n-1}\| > r\}) \\ &= \mathbb{P}(A_n)\mathbb{P}(\|x_n + S_N - S_n\| > r) \\ &= \mathbb{P}(A_n)\mathbb{P}(\|S_N - S_{n-1}\| > r) \\ &\le 2\mathbb{P}(A_n)\mathbb{P}(\|S_N\| > r),\end{aligned}$$

where we use Lévy's inequality for the final step.

Summing over $n = 1, \dots, N$ and using Lévy's inequality again,

$$\begin{aligned}\mathbb{P}(\|S_N\| > 2r) &= \sum_{n=1}^{N} \mathbb{P}(A_n \cap \{\|S_N\| > 2r\}) \\ &\le 2\sum_{n=1}^{N} \mathbb{P}(A_n)\mathbb{P}(\|S_N\| > r) \\ &= 2\mathbb{P}\left(\max_{1\le n\le N} \|S_n\| > r\right)\mathbb{P}(\|S_N\| > r) \\ &\le 4\mathbb{P}(\|S_N\| > r)^2,\end{aligned}$$

which is the inequality that we sought to prove. □

This lemma enables us to prove the Kahane-Khintchine inequalities by computing $\kappa_{p,1}$ for $p > 1$ and using Hölder's inequality for the remaining cases.

Theorem 2.3.11 (Kahane-Khintchine inequalities)**.** *For all $p, q \in [1, \infty)$, there exists a constant $\kappa_{p,q}$ such that for any Banach space X and all finite sequences $(x_n)_{n=1}^N$ in X,*

$$\left\|\sum_{n=1}^{N} \varepsilon_n x_n\right\|_{L^p(\Omega;X)} \le \kappa_{p,q}\left\|\sum_{n=1}^{N} \varepsilon_n x_n\right\|_{L^q(\Omega;X)},$$

where $(\varepsilon_n)_{n=1}^N$ is a Rademacher sequence on a probability space Ω.

Proof. By Corollary 2.3.8, it suffices to consider a real Rademacher sequence $(r_n)_{n=1}^N$, and by Hölder's inequality, it suffices to consider the case where $p > 1$ and $q = 1$. Fix a finite sequence $(x_n)_{n=1}^N$ in X, let $Y_n = r_n x_n$ for all $n = 1, \dots, N$, and let $S_N = \sum_{n=1}^N Y_n$. By scaling and using homogeneity, we can suppose without loss of generality that $\mathbb{E}\|S_N\| = 1$. Now, it remains to show that

$$\mathbb{E}\|S_N\|^p \le \kappa_{p,1}.$$

Let $j \in \mathbb{N}$ be such that $2^{j-1} < p \leq 2^j$, which is unique. Successively applying the previous lemma for $r > 0$, we find that

$$\mathbb{P}(\|S_N\| > 2^j r) \leq 4^{2^j - 1}(\mathbb{P}(\|S_N\| > r))^{2^j}.$$

By Markov's inequality, $\mathbb{P}(\|S_N\| > r) \leq \frac{1}{r}\mathbb{E}\|S_N\| = \frac{1}{r}$. Then, we can compute

$$\begin{aligned}
\mathbb{E}\|S_N\|^p &= \int_0^\infty p t^{p-1} \mathbb{P}(\|S_N\| > t)\,\mathrm{d}t \\
&= 2^{jp} \int_0^\infty p r^{p-1} \mathbb{P}(\|S_N\| > 2^j r)\,\mathrm{d}r \\
&\leq 2^{jp} 4^{2^j - 1} p \int_0^\infty r^{p-1} \mathbb{P}(\|S_N\| > r)^{2^j}\,\mathrm{d}r \\
&\leq (2p)^p 4^{2p-1} p \int_0^\infty r^{p-1} \mathbb{P}(\|S_N\| > r)^p\,\mathrm{d}r \\
&\leq (2p)^p 4^{2p-1} p \int_0^\infty \mathbb{P}(\|S_N\| > r)\,\mathrm{d}r \\
&= (2p)^p 4^{2p-1} p \mathbb{E}\|S_N\| \\
&= (2p)^p 4^{2p-1} p,
\end{aligned}$$

which is a constant depending only on p. □

The content of these inequalities is that, up to some constant, the L^p-norm of a Rademacher sum is independent of the choice of $p \in [1, \infty)$. This will be enormously useful to us when studying the plethora of Banach space properties characterized by the behavior of Rademacher sums: the randomized UMD properties, for example, but also K-convexity, type and cotype, and R-boundedness of operator families.

Before proceeding to the study of UMD spaces, we make note of a useful corollary of the Kahane-Khintchine inequalities in the Hilbert space case, which matches Khintchine's original inequality for scalars in [Khi23].

Corollary 2.3.12 (Khintchine's inequality)**.** *For all $p \in [1, \infty)$, there exist constants $0 < A_p \leq B_p < \infty$ such that for any Hilbert space H and all finite sequences $(h_n)_{n=1}^N$ in H,*

$$A_p \left(\sum_{n=1}^N \|h_n\|_H^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{n=1}^N \varepsilon_n h_n \right\|_{L^p(\Omega; H)} \leq B_p \left(\sum_{n=1}^N \|h_n\|_H^2 \right)^{\frac{1}{2}},$$

where $(\varepsilon_n)_{n=1}^N$ is a Rademacher sequence on a probability space Ω.

Proof. As $(\varepsilon_n)_{n=1}^N$ is a sequence of independent, mean zero random variables,

$$\mathbb{E}[\varepsilon_m \overline{\varepsilon}_n] = \mathbb{E}[\varepsilon_m]\mathbb{E}[\overline{\varepsilon}_m] = 0 \cdot 0 = 0$$

for any $m \neq n$, so $(\varepsilon_n)_{n=1}^N$ is orthogonal in $L^2(\Omega)$. For $p = 2$, it follows that

$$\left\| \sum_{n=1}^N \varepsilon_n h_n \right\|_{L^2(\Omega;H)}^2 = \sum_{n=1}^N \|h_n\|_H^2.$$

For $p \neq 2$, the desired inequalities follow from the $p = 2$ case and the Kahane-Khintchine inequalities. □

This concludes our study of martingales in Banach spaces and Rademacher sequences. With the probabilistic machinery that we have constructed, we are now prepared to treat the properties of UMD, UMD$^+$, and UMD$^-$ spaces, which we discuss in the next chapter.

Chapter 3

UMD and Randomized UMD Spaces

After our discussion of martingales in Banach spaces and Rademacher sequences, we can return to UMD and randomized UMD spaces. In this chapter, we will review the definitions and fundamental results about the UMD and randomized UMD properties. Then, we discuss basic constructions of UMD and randomized UMD spaces with several important examples, including the spaces c_0, ℓ^∞, ℓ^1, and L^p. Finally, we mention an application to the R-boundedness of conditional expectation operators.

3.1 Initialization

In this section, we restate the definitions of the UMD and randomized UMD properties. We also discuss the major results mentioned in the introduction, including the proof that a Banach space is a UMD space if and only if it is a UMD$^+$ space and a UMD$^-$ space. Let us begin with a review of the definitions, now accounting for p-independence given by Theorems 1.1.2 and 1.1.6.

Definition 1.1.1 (UMD)**.** *A Banach space X is called a* UMD space *if for some (equivalently, for all) $p \in (1, \infty)$, there exists a constant $\beta \in (0, \infty)$ such that for any $L^p(S;X)$-martingale difference sequence $(df_n)_{n=1}^N$ on a σ-finite measure space $(S, \mathcal{A}, \mu)$ and scalars $|\epsilon_n| = 1$, $n = 1, \ldots, N$, we have*

$$\left\| \sum_{n=1}^N \epsilon_n df_n \right\|_{L^p(S;X)} \leq \beta \left\| \sum_{n=1}^N df_n \right\|_{L^p(S;X)}.$$

Definition 1.1.4 (UMD$^+$)**.** *A Banach space* X *is called a* UMD$^+$ space *if for some (equivalently, for all)* $p \in (1,\infty)$*, there exists a constant* $\beta^+ \in (0,\infty)$ *such that for any* $L^p(S;X)$*-martingale difference sequence* $(df_n)_{n=1}^N$ *on a* σ*-finite measure space* $(S,\mathcal{A},\mu)$*, we have*

$$\left\|\sum_{n=1}^N \varepsilon_n df_n\right\|_{L^p(S\times\Omega;X)} \le \beta^+ \left\|\sum_{n=1}^N df_n\right\|_{L^p(S;X)}$$

where $(\varepsilon_n)_{n=1}^N$ *is a Rademacher sequence on a probability space* Ω*.*

Definition 1.1.5 (UMD$^-$)**.** *A Banach space* X *is called a* UMD$^-$ space *if for some (equivalently, for all)* $p \in (1,\infty)$*, there exists a constant* $\beta^+ \in (0,\infty)$ *such that for any* $L^p(S;X)$*-martingale difference sequence* $(df_n)_{n=1}^N$ *on a* σ*-finite measure space* $(S,\mathcal{A},\mu)$*, we have*

$$\left\|\sum_{n=1}^N df_n\right\|_{L^p(S;X)} \le \beta^- \left\|\sum_{n=1}^N \varepsilon_n df_n\right\|_{L^p(S\times\Omega;X)}$$

where $(\varepsilon_n)_{n=1}^N$ *is a Rademacher sequence on a probability space* Ω*.*

Although the definition of UMD appears to more closely resemble that of UMD$^+$ than that of UMD$^-$, its defining inequality can be reversed. If $(df_n)_{n=1}^N$ is a martingale difference sequence, $(\epsilon_n df_n)_{n=1}^N$ is as well. For a UMD space X, this gives the reverse inequality

$$\left\|\sum_{n=1}^N df_n\right\|_{L^p(S;X)} \le \beta \left\|\sum_{n=1}^N \epsilon_n df_n\right\|_{L^p(S;X)},$$

which resembles the defining inequality for the UMD$^-$ property.

As we discussed in the introduction, each condition is satisfied for all $p \in (1,\infty)$ if it is satisfied by some, a result of [Mau75, Bur81] for UMD and [Gar90, Theorem 4.1] for UMD$^+$ and UMD$^-$. For UMD spaces, this can be shown by reducing to martingales adapted to *Walsh-Paley filtrations*: those in which each set in $\mathcal{F}_n$ is a union of atoms of measure 2^{-n}. It is unknown whether the same reduction can be done for UMD$^+$ and UMD$^-$. See [CG21, Theorem 1.4] for a recent partial result.

Nevertheless, we still care to distinguish between different choices of p because they lead to different optimal constants. If a Banach space X is UMD, we denote by $\beta_p(X)$ the infimum over all admissible β for the particular choice of p. The same holds for $\beta_p^+(X)$ for UMD$^+$ and $\beta_p^-(X)$ for UMD$^-$.

Now, we proceed to restate and finally prove Theorem 1.1.3.

Theorem 1.1.3. *A Banach space X is a UMD space if and only if it is a UMD$^+$ space and a UMD$^-$ space.*

Proof. Suppose that X is a UMD space and fix $p \in (1, \infty)$, so that

$$\frac{1}{\beta_p(X)} \left\| \sum_{n=1}^{N} df_n \right\|_{L^p(S;X)} \leq \left\| \sum_{n=1}^{N} \epsilon_n df_n \right\|_{L^p(S;X)} \leq \beta_p(X) \left\| \sum_{n=1}^{N} df_n \right\|_{L^p(S;X)}$$

for any $L^p(S;X)$-martingale difference sequence $(df_n)_{n=1}^N$ and any sequence of scalars $(\epsilon_n)_{n=1}^N$ with $|\epsilon| = 1$ for $n = 1, \dots, N$. In particular, for a Rademacher sequence $(\varepsilon_n)_{n=1}^N$ on a probability space Ω,

$$\frac{1}{\beta_p(X)} \left\| \sum_{n=1}^{N} df_n \right\|_{L^p(S;X)} \leq \left\| \sum_{n=1}^{N} \varepsilon_n(\omega) df_n \right\|_{L^p(S;X)} \leq \beta_p(X) \left\| \sum_{n=1}^{N} df_n \right\|_{L^p(S;X)}$$

for any $\omega \in \Omega$. Taking the $L^p(\Omega)$-norm, this yields the randomized UMD properties with constants $\beta^+ = \beta^- = \beta_p(X)$.

For the converse, suppose that X is UMD$^+$ and UMD$^-$, then fix $p \in (1, \infty)$. Let $(df_n)_{n=1}^N$ be an $L^p(S;X)$-martingale difference sequence and let $(\epsilon_n)_{n=1}^N$ be a sequence of scalars with $|\epsilon| = 1$ for $n = 1, \dots, N$. Then, using for the equality that $(\epsilon_n \varepsilon_n)_{n=1}^N$ and $(\varepsilon_n)_{n=1}^N$ are identically distributed,

$$\begin{aligned}
\frac{1}{\beta_p^-(X)} \left\| \sum_{n=1}^{N} \epsilon_n df_n \right\|_{L^p(S;X)} &\leq \left\| \sum_{n=1}^{N} \epsilon_n \varepsilon_n df_n \right\|_{L^p(S\times\Omega;X)} \\
&= \left\| \sum_{n=1}^{N} \varepsilon_n df_n \right\|_{L^p(S\times\Omega;X)} \\
&\leq \beta_p^+(X) \left\| \sum_{n=1}^{N} df_n \right\|_{L^p(S;X)},
\end{aligned}$$

so X is a UMD space with constant $\beta = \beta_p^-(X)\beta_p^+(X)$. □

It is easy to see from the proof of Theorem 1.1.3 that

$$\beta_p^-(X), \beta_p^+(X) \leq \beta_p(X) \leq \beta_p^-(X)\beta_p^+(X).$$

We mentioned in the introduction that Theorem 1.1.3 can be used to establish a quantitative relationship between UMD constants and the operator norm $\hbar_p(X) := \|H\|_{L^p(\mathbb{R};X)\to L^p(\mathbb{R};X)}$ of the Hilbert transform H. As shown in [Gar86], it holds that

$$\hbar_p(X) \leq \beta_p^+(X)\beta_p^-(X).$$

If the conjecture that UMD and UMD$^+$ are equivalent is true, one possible proof may involve eliminating the dependence on $\beta_p^-(X)$ in the inequality above.

During the rest of this chapter, we will discuss constructions and properties which follow directly from the definitions given in this section.

3.2 Properties

The remainder of this chapter concerns various constructions, properties, and counterexamples of both UMD$^+$ and UMD$^-$ spaces. We begin this section with some basic constructions, then proceed to show that the underlying scalar field does not qualitatively affect the UMD$^+$ or UMD$^-$ properties, prove a duality result, and finally discuss finite representability.

3.2.1 Basic constructions

We begin this section with the most basic constructions of UMD$^+$ and UMD$^-$ spaces. First, we prove that all Hilbert spaces are UMD spaces (and thus both UMD$^+$ and UMD$^-$ spaces).

Proposition 3.2.1. *Every Hilbert space H is a UMD space with $\beta_2(H) = 1$.*

Proof. Let $(df_n)_{n=1}^N$ be an H-valued L^p-martingale difference sequence on a σ-finite measure space $(S, \mathcal{A}, \mu)$. By Proposition 2.2.4, $(df_n)_{n=1}^N$ is mutually orthogonal and

$$\left\|\sum_{n=1}^N df_n\right\|_{L^2(S;H)} = \left(\sum_{n=1}^N \|df_n\|_{L^2(S;H)}^2\right)^{\frac{1}{2}}.$$

As $(df_n)_{n=1}^N$ is a martingale difference sequence, $(\epsilon_n df_n)_{n=1}^N$ is as well, so the same identity holds. Therefore,

$$\left\|\sum_{n=1}^N \epsilon_n df_n\right\|_{L^2(S;H)} = \left(\sum_{n=1}^N \|df_n\|_{L^2(S;H)}^2\right)^{\frac{1}{2}} = \left\|\sum_{n=1}^N df_n\right\|_{L^2(S;H)},$$

which shows that H is a UMD space and $\beta_2(H) = 1$. □

In particular, the scalar fields $\mathbb{R}$ and $\mathbb{C}$ are UMD spaces. Once we provide examples of UMD$^+$ and UMD$^-$ spaces which are *not* Hilbert spaces, it will be reasonable to think of the UMD and randomized UMD properties as particular generalizations of Hilbert spaces to include a wider class of Banach spaces. The motivation for this perspective comes from the discussion following Proposition 2.2.4.

Next, we prove that both UMD$^+$ and UMD$^-$ (thus also UMD) are preserved under isomorphisms and the taking of closed subspaces. These two basic methods for constructing new UMD$^+$ and UMD$^-$ spaces from existing ones follow directly from the definitions of the randomized UMD properties.

Proposition 3.2.2. *If X is a UMD$^+$ (resp. UMD$^-$) space and Y is isomorphic to X via $J : X \to Y$, then Y is a UMD$^+$ (resp. UMD$^-$) space with*

$$\beta_p^{\pm}(Y) \leq \|J\|\|J^{-1}\|\,\beta_p^{\pm}(X).$$

Proof. Identifying Y-valued martingales with X-valued martingales via the isomorphism J, this follows directly from the definitions of the randomized UMD properties. □

Proposition 3.2.3. *If X is a UMD$^+$ (resp. UMD$^-$) space and Y is a closed subspace of X, then Y is also a UMD$^+$ (resp. UMD$^-$) space with $\beta_p^{\pm}(Y) \leq \beta_p^{\pm}(X)$.*

Proof. Any Y-valued martingale is also an X-valued martingale, so this follows directly from the definitions of the randomized UMD properties. □

3.2.2 Underlying scalar field

So far, we have used for the UMD$^+$ and UMD$^-$ properties Rademacher sequences which take values in the underlying scalar field of the relevant Banach space. As we will see, these properties are actually independent of the choice of scalar field. Using real Rademacher sequences is sufficient to show that a complex Banach space is UMD$^+$ or UMD$^-$.

For a complex Banach space X, denote by $X_{\mathbb{R}}$ the Banach space obtained by restricting scalar multiplication to real numbers. As we will see, the UMD$^+$ and UMD properties hold for X if and only if they hold for $X_{\mathbb{R}}$. Corollary 2.3.8, which was a consequence of Kahane's contraction principle, makes the proof simple.

Proposition 3.2.4. *Let X be a complex Banach space and let $p \in (1,\infty)$. X is a UMD$^+$ (resp. UMD$^-$) space if and only if $X_{\mathbb{R}}$ is a UMD$^+$ (resp. UMD$^-$) space. In that case,*

$$\frac{2}{\pi}\beta_p^{\pm}(X_{\mathbb{R}}) \leq \beta_p^{\pm}(X) \leq \frac{\pi}{2}\beta_p^{\pm}(X_{\mathbb{R}}).$$

Proof. If X is a UMD$^+$ space, then for a real Rademacher sequence $(r_n)_{n=1}^N$ and an $L^p(S;X)$-martingale difference sequence $(df_n)_{n=1}^N$,

$$\left\|\sum_{n=1}^N r_n df_n\right\|_{L^p(S\times\Omega;X)} \leq \frac{\pi}{2}\left\|\sum_{n=1}^N \varepsilon_n df_n\right\|_{L^p(S\times\Omega;X)} \leq \frac{\pi}{2}\beta_p^+(X)\left\|\sum_{n=1}^N df_n\right\|_{L^p(S;X)},$$

using Corollary 2.3.8, so $X_{\mathbb{R}}$ is a UMD$^+$ space with $\beta_p^+(X_{\mathbb{R}}) \leq \frac{\pi}{2}\beta_p^+(X)$.

If $X_{\mathbb{R}}$ is a UMD$^+$ space, then for a complex Rademacher sequence $(\varepsilon_n)_{n=1}^N$ and an $L^p(S;X)$-martingale difference sequence $(df_n)_{n=1}^N$,

$$\left\|\sum_{n=1}^N \varepsilon_n df_n\right\|_{L^p(S\times\Omega;X)} \leq \frac{\pi}{2}\left\|\sum_{n=1}^N r_n df_n\right\|_{L^p(S\times\Omega;X)} \leq \frac{\pi}{2}\beta_p^+(X_{\mathbb{R}})\left\|\sum_{n=1}^N df_n\right\|_{L^p(S;X)},$$

using Corollary 2.3.8, so X is a UMD$^+$ space with $\beta_p^+(X) \leq \frac{\pi}{2}\beta_p^+(X_{\mathbb{R}})$.

If X is a UMD$^-$ space, then for a real Rademacher sequence $(r_n)_{n=1}^N$ and an $L^p(S;X)$-martingale difference sequence $(df_n)_{n=1}^N$,

$$\left\|\sum_{n=1}^N df_n\right\|_{L^p(S;X)} \leq \beta_p^-(X)\left\|\sum_{n=1}^N \varepsilon_n df_n\right\|_{L^p(S\times\Omega;X)} \leq \frac{\pi}{2}\beta_p^-(X)\left\|\sum_{n=1}^N r_n df_n\right\|_{L^p(S\times\Omega;X)},$$

using Corollary 2.3.8, so $X_{\mathbb{R}}$ is a UMD$^-$ space with $\beta_p^-(X_{\mathbb{R}}) \leq \frac{\pi}{2}\beta_p^-(X)$.

If $X_{\mathbb{R}}$ is a UMD$^-$ space, then for a complex Rademacher sequence $(\varepsilon_n)_{n=1}^N$ and an $L^p(S;X)$-martingale difference sequence $(df_n)_{n=1}^N$,

$$\left\|\sum_{n=1}^N df_n\right\|_{L^p(S;X)} \leq \beta_p^-(X)\left\|\sum_{n=1}^N r df_n\right\|_{L^p(S\times\Omega;X)} \leq \frac{\pi}{2}\beta_p^-(X)\left\|\sum_{n=1}^N \varepsilon_n df_n\right\|_{L^p(S\times\Omega;X)},$$

using Corollary 2.3.8, so X is a UMD$^-$ space with $\beta_p^-(X) \leq \frac{\pi}{2}\beta_p^-(X_{\mathbb{R}})$. □

This equivalence will later allow us to use the UMD$^+$ or UMD$^-$ property of $X_{\mathbb{R}}$ when we are given that X is a UMD$^+$ or UMD$^-$ space. This will be especially useful when studying K-convexity, which we define in a way that is independent of the choice of scalar field $\mathbb{K}$.

Note that this equivalence also holds for the UMD property and its associated constants. We can show the same relationship by considering UMD$^+$ and UMD$^-$ separately, then using that the randomized UMD properties are together equivalent to UMD by Theorem 1.1.3. In fact, if X is a UMD space, we can consider only real $\epsilon_1, \ldots, \epsilon_N$ in the definition to conclude that $X_{\mathbb{R}}$ is also a UMD space and $\beta_p(X_{\mathbb{R}}) \leq \beta_p(X)$ without the factor of $\frac{\pi}{2}$. The second inequality, $\beta_p(X) \leq \frac{\pi}{2}\beta_p(X_{\mathbb{R}})$, remains unchanged.

3.2.3 Duality

The first distinct result about UMD$^+$ and UMD$^-$ spaces is that they are *almost* dual to each other. [Gar90, Theorem 3.1] gives us that the dual and predual (if it exists) of a UMD$^+$ space X are UMD$^-$ spaces. If X is also a UMD$^-$ space (hence a UMD space), then the dual and predual (if it exists) are UMD spaces as well.

Proposition 3.2.5. *If X is a UMD$^+$ space, then its dual X^* is a UMD$^-$ space with $\beta^-_{p'}(X^*) \leq \beta^+_p(X)$. If X^* is a UMD$^+$ space, then its predual X is a UMD$^-$ space with $\beta^-_{p'}(X) \leq \beta^+_p(X^*)$.*

Proof. Suppose that X is a UMD$^+$ space. Let $(\phi_n)_{n=1}^N$ be an $L^{p'}(S;X^*)$-martingale adapted to the filtration $(\mathcal{F}_n)_{n=1}^N$. Fix an arbitrary $f \in L^p(S,\mathcal{F}_N;X)$ such that $\|f\|_{L^p(S;X)} = 1$, then define the $L^p(S;X)$-martingale $(f_n)_{n=1}^N$ by $f_n = \mathbb{E}[f \mid \mathcal{F}_n]$ so that $f_N = f$. For $1 \leq m < n \leq N$, we can compute

$$\mathbb{E}\left\langle df_m, d\phi_n\right\rangle = \mathbb{E}\mathbb{E}[\left\langle df_m, d\phi_n\right\rangle \mid \mathcal{F}_{n-1}] = \mathbb{E}\left\langle df_m, \mathbb{E}[d\phi_n \mid \mathcal{F}_{n-1}]\right\rangle = 0,$$

and $\mathbb{E}\left\langle df_m, d\phi_n\right\rangle = 0$ for $1 \leq n < m \leq N$ by a similar argument.

Therefore,

$$\begin{aligned}
\left|\mathbb{E}\left\langle f, \sum_{n=1}^{N} d\phi_n \right\rangle\right| &= \left|\mathbb{E}\left\langle \sum_{m=1}^{N} df_m, \sum_{n=1}^{N} d\phi_n \right\rangle\right| \\
&= \left|\mathbb{E}\left\langle \sum_{m=1}^{N} \varepsilon_m df_m, \sum_{n=1}^{N} \varepsilon_n d\phi_n \right\rangle\right| \\
&\leq \left\|\sum_{m=1}^{N} \varepsilon_m df_m\right\|_{L^p(S\times\Omega;X)} \left\|\sum_{n=1}^{N} \varepsilon_n d\phi_n\right\|_{L^{p'}(S\times\Omega;X^*)} \\
&\leq \beta_p^+(X) \left\|\sum_{m=1}^{N} df_m\right\|_{L^p(S;X)} \left\|\sum_{n=1}^{N} \varepsilon_n d\phi_n\right\|_{L^{p'}(S\times\Omega;X^*)} \\
&= \beta_p^+(X)\|f\|_{L^p(S;X)} \left\|\sum_{n=1}^{N} \varepsilon_n d\phi_n\right\|_{L^{p'}(S\times\Omega;X^*)} \\
&= \beta_p^+(X) \left\|\sum_{n=1}^{N} \varepsilon_n d\phi_n\right\|_{L^{p'}(S\times\Omega;X^*)}.
\end{aligned}$$

Now, $L^p(S, \mathcal{F}_N; X)$ is norming for $L^{p'}(S, \mathcal{F}_N; X^*)$, so we can take the supremum over all $f \in L^p(S, \mathcal{F}_N; X)$ with $\|f\|_{L^p(S;X)} = 1$ to obtain

$$\left\|\sum_{n=1}^{N} d\phi_n\right\|_{L^{p'}(S;X^*)} \leq \beta_p^+(X) \left\|\sum_{n=1}^{N} \varepsilon_n d\phi_n\right\|_{L^{p'}(S\times\Omega;X^*)}.$$

That is, X^* is a UMD$^-$ space with $\beta_{p'}^-(X^*) \leq \beta_p^+(X)$.

Now, suppose instead that X^* is a UMD$^+$ space. Applying the result above, we find that X^{**} is a UMD$^-$ space with $\beta_{p'}^-(X^{**}) \leq \beta_p^+(X^*)$. X is isometric to a closed subspace of X^{**}, so by Propositions 3.2.2 and 3.2.3, X is also a UMD$^-$ space with $\beta_{p'}^-(X) \leq \beta_{p'}^-(X^{**}) \leq \beta_p^+(X^*)$. □

This proposition is weaker than we might like it to be, as the dual and predual of a UMD$^-$ space are not necessarily UMD$^+$. As we will later see in Examples 3.3.2 and 3.3.3, ℓ^1 is a UMD$^-$ space, but its predual c_0 and dual ℓ^∞ are neither UMD$^-$ nor UMD$^+$.

If we require that X is also a UMD$^-$ space (hence UMD), this problem is solved: the following proposition shows that the dual and predual (if it exists) of a UMD space are also UMD spaces. The proof is almost identical to that of the previous proposition.

Proposition 3.2.6. *A Banach space X is a UMD space if and only if its dual X^* is also a UMD space. In that case, $\beta_{p'}(X^*) = \beta_p(X)$.*

Proof. Suppose that X is a UMD space. Let $(\phi_n)_{n=1}^N$ be an $L^{p'}(S;X^*)$-martingale adapted to the filtration $(\mathcal{F}_n)_{n=1}^N$. Fix an arbitrary $f \in L^p(S,\mathcal{F}_N;X)$ such that $\|f\|_{L^p(S;X)} = 1$, then define the $L^p(S;X)$-martingale $(f_n)_{n=1}^N$ by $f_n = \mathbb{E}[f \mid \mathcal{F}_n]$ so that $f_N = f$. For $1 \le m < n \le N$, we can compute

$$\mathbb{E}\langle df_m, d\phi_n\rangle = \mathbb{E}\mathbb{E}[\langle df_m, d\phi_n\rangle \mid \mathcal{F}_{n-1}] = \mathbb{E}\langle df_m, \mathbb{E}[d\phi_n \mid \mathcal{F}_{n-1}]\rangle = 0,$$

and $\mathbb{E}\langle df_m, d\phi_n\rangle = 0$ for $1 \le n < m \le N$ by a similar argument.

Therefore,

$$\begin{aligned}
\left|\mathbb{E}\left\langle f, \sum_{n=1}^N \epsilon_n d\phi_n\right\rangle\right| &= \left|\mathbb{E}\left\langle \sum_{m=1}^N df_m, \sum_{n=1}^N \epsilon_n d\phi_n\right\rangle\right| \\
&= \left|\mathbb{E}\left\langle \sum_{m=1}^N \epsilon_m df_m, \sum_{n=1}^N d\phi_n\right\rangle\right| \\
&\le \left\|\sum_{m=1}^N \epsilon_m df_m\right\|_{L^p(S;X)} \left\|\sum_{n=1}^N d\phi_n\right\|_{L^{p'}(S;X^*)} \\
&\le \beta_p(X)\left\|\sum_{m=1}^N df_m\right\|_{L^p(S;X)} \left\|\sum_{n=1}^N d\phi_n\right\|_{L^{p'}(S;X^*)} \\
&= \beta_p(X)\|f\|_{L^p(S;X)}\left\|\sum_{n=1}^N d\phi_n\right\|_{L^{p'}(S;X^*)} \\
&= \beta_p(X)\left\|\sum_{n=1}^N d\phi_n\right\|_{L^{p'}(S;X^*)}.
\end{aligned}$$

Now, $L^p(S,\mathcal{F}_N;X)$ is norming for $L^{p'}(S,\mathcal{F}_N;X^*)$, so we can take the supremum over all $f \in L^p(S, \mathcal{F}_N; X)$ with $\|f\|_{L^p(S;X)} = 1$ to obtain

$$\left\| \sum_{n=1}^{N} \epsilon_n d\phi_n \right\|_{L^{p'}(S;X^*)} \leq \beta_p(X) \left\| \sum_{n=1}^{N} d\phi_n \right\|_{L^{p'}(S;X^*)}.$$

That is, X^* is a UMD space with $\beta_{p'}(X^*) \leq \beta_p(X)$.

This result applied to X^* and X^{**} gives that X^{**} is also a UMD space and $\beta_p(X^{**}) \leq \beta_{p'}(X^*)$. X is isometric to a closed subspace of X^{**}, so by Propositions 3.2.2 and 3.2.3, it is also a UMD space with $\beta_p(X) \leq \beta_p(X^{**}) \leq \beta_{p'}(X^*)$. Combining this result with the first inequality in this paragraph, we find that $\beta_{p'}(X^*) = \beta_p(X)$. □

3.2.4 Finite representability

In this subsection, we briefly discuss the finite representability of Banach spaces in one another as well as the notion of super-properties. First, we define finite representability, which was first described in [Jam72a, Jam72b].

Definition 3.2.7 (Finitely representable)**.** *A Banach space Y is said to be finitely representable in a Banach space X if, for every $\epsilon > 0$ and every finite-dimensional subspace $Y_0 \subseteq Y$, there exists a finite-dimensional subspace $X_0 \subseteq X$ and a linear isomorphism $J : Y_0 \to X_0$ such that*

$$\|J\|\|J^{-1}\| \leq 1 + \epsilon.$$

With this definition, we call a property which a Banach space may or may not satisfy (e.g. reflexivity, UMD, K-convexity) a *super-property* if it is preserved under finite representability. That is, if a Banach space X satisfying the property and another Banach space Y being finitely representable in X implies that Y also satisfies the property, it is a super-property.

By Lemma 2.2.5, we can estimate arbitrary martingales by simple martingales supported on a finite measure set. Therefore, when treating the UMD$^+$ and UMD$^-$ properties for a Banach space X, we need only consider simple martingales whose images are contained in finite-dimensional subspaces of X. For this reason, both UMD$^+$ and UMD$^-$ (and thus UMD as well) are super-properties.

3.3 Fundamental examples

Having demonstrated several basic constructions of new UMD^+ and UMD^- spaces from existing ones, we continue by mentioning some important examples of UMD^+ and UMD^- spaces. In the first subsection, we test whether some sequence spaces (c_0, ℓ^∞, and ℓ^1) satisfy the UMD^+ or UMD^- properties. In the second, we prove that certain L^p spaces are UMD^+ and UMD^- spaces.

3.3.1 Sequence spaces

Our first fundamental example is actually a pair of counterexamples. We now prove that the sequence spaces c_0 and ℓ^∞ are neither UMD^+ not UMD^-. We begin with an estimate for the randomized UMD constants of the finite-dimensional space $\ell^\infty_{2^N}$, which we will proceed to embed in c_0 and ℓ^∞. The estimate for $\beta_p^-(\ell^\infty_{2^N})$ is due to [Gar90, Section 3], while that for $\beta_p^+(\ell^\infty_{2^N})$ is an adaptation of the same argument. For discussion of the tightest asymptotics for these constants, see [PW98] and [Wen05].

Lemma 3.3.1. *Let $N \in \mathbb{N}$ and $p \in (1,\infty)$. Then,*

$$\beta_p^+(\ell^\infty_{2^N}) \geq c_p^+\sqrt{N}, \quad \beta_p^-(\ell^\infty_{2^N}) \geq c_p^-\sqrt{N}$$

for some constants $c_p^+, c_p^- > 0$ depending only on p.

Proof. Let $D = \{-1,1\}^N$ equipped with the uniform probability measure μ. For $n = 1,\dots,N$, consider the functions on $D \times D$ given by

$$d_n(s,t) = \begin{cases} +1 & \text{if } s_j = t_j \text{ for } j \leq n \\ -1 & \text{if } s_j = t_j \text{ for } j < n \text{ and } s_n \neq t_n \\ 0 & \text{if } s_j \neq t_j \text{ for some } j < n. \end{cases}$$

For any fixed $t \in D$, we have that $(s \mapsto d_n(s,t))_{n=1}^N$ is an $L^p(D)$-martingale difference sequence. Therefore, $(s \mapsto (d_n(s,t))_{t\in D})_{n=1}^N$ is an $L^p(D;\ell^\infty(D))$-martingale difference sequence.

Let $(\varepsilon_n)_{n=1}^N$ be a Rademacher sequence. Then, we can estimate

$$\left|\sum_{n=1}^N \varepsilon_n d_n(s,t)\right| \leq \sup_{1\leq M\leq N}\left|\sum_{n=1}^M \varepsilon_n + \varepsilon_{M+1}\mathbb{1}_{\{M<N\}}\right| \leq S_N^* + 1,$$

where $S_n = \sum_{j=1}^{n} \varepsilon_j$.

Taking $L^p(\Omega \times D; \ell^\infty(D))$-norms (with respect to the variable s),

$$\left\| \sum_{n=1}^{N} \varepsilon_n d_n(s, \cdot) \right\|_{L^p(\Omega \times D; \ell^\infty(D))} \leq \|S_N^*\|_{L^p(\Omega)} + 1 \leq p' \|S_N\|_{L^p(\Omega)} + 1 \leq C_p \sqrt{N},$$

for some constant $C_p > 0$, where we use Doob's maximal inequality followed by Khintchine's inequality.

Now, using that $d_n(s,s) = 1$, the definition of the UMD$^-$ constant, and the previous inequality, we find that

$$\begin{aligned} N &= \inf_{s \in D} \left| \sum_{n=1}^{N} d_n(s,s) \right| \\ &\leq \left\| \sum_{n=1}^{N} d_n(s, \cdot) \right\|_{L^p(D; \ell^\infty(D))} \\ &\leq \beta_p^-(\ell^\infty(D)) \left\| \sum_{n=1}^{N} \varepsilon_n d_n(s, \cdot) \right\|_{L^p(\Omega \times D; \ell^\infty(D))} \\ &\leq \beta_p^-(\ell^\infty(D)) C_p \sqrt{N}, \end{aligned}$$

so $\beta_p^-(\ell^\infty(D)) \geq C_p^{-1} \sqrt{N}$. Writing $c_p^- = C_p^{-1}$ and identifying $\ell^\infty(D) \cong \ell^\infty_{2^N}$, Proposition 3.2.2 gives us the desired result.

Similarly, since $d_n(s,s) = 1$, we have that

$$\left\| \sum_{n=1}^{N} \varepsilon_n d_n(s, \cdot) \right\|_{L^p(\Omega \times D; \ell^\infty(D))} \geq \|S_N\|_{L^p(\Omega)} \geq c_p \sqrt{N},$$

for some constant $c_p > 0$, where we once again use Khintchine's inequality. As $(\varepsilon_n(-1)^n)_{n=1}^N$ is still a Rademacher sequence, we can replace $\varepsilon_n d_n(s, \cdot)$ with $\varepsilon_n(-1)^n d_n(s, \cdot)$ in these inequalities.

Now, we claim that for $s, t \in D$,

$$\left| \sum_{n=1}^{N} (-1)^n d_n(s,t) \right| \leq 2.$$

For the proof, let $M \in \{1, \dots, N\}$ be the maximum such that $s_j = t_j$ for $j \leq M$. By construction of $(d_n)_{n=1}^N$, this implies that $d_n(s,t) = 1$ for $n \leq M$, $d_{M+1}(s,t) = -1$ if $M < N$, and $d_n(s,t) = 0$ for $n > M+1$. Therefore,

$$\begin{aligned}\left|\sum_{n=1}^{N}(-1)^n d_n(s,t)\right| &= \left|\sum_{n=1}^{M}(-1)^n \cdot 1 + (-1)^{M+1} \cdot (-1) \cdot \mathbb{1}_{\{M<N\}}\right| \\ &= \left|(-1)\mathbb{1}_{\{M \text{ odd}\}} + (-1)^M \mathbb{1}_{\{M<N\}}\right| \\ &\leq 2.\end{aligned}$$

Using our previous two inequalities and the definition of the UMD$^+$ constant,

$$\begin{aligned}c_p\sqrt{N} &\leq \left\|\sum_{n=1}^{N} \varepsilon_n (-1)^n d_n(s,\cdot)\right\|_{L^p(\Omega\times D;\ell^\infty(D))} \\ &\leq \beta_p^+(\ell^\infty(D))\left\|\sum_{n=1}^{N}(-1)^n d_n(s,\cdot)\right\|_{L^p(D;\ell^\infty(D))} \\ &\leq \beta_p^+(\ell^\infty(D)) \sup_{s,t\in D}\left|\sum_{n=1}^{N}(-1)^n d_n(s,t)\right| \\ &\leq 2\beta_p^+(\ell^\infty(D)),\end{aligned}$$

so $\beta_p^+(\ell^\infty(D)) \geq c_p\sqrt{N}/2$. Writing $c_p^+ = c_p/2$ and identifying $\ell^\infty(D) \cong \ell^\infty_{2^N}$, Proposition 3.2.2 gives us the desired result. □

Embedding $\ell^\infty_{2^N}$ in c_0 and ℓ^∞, we can use these bounds on $\beta_p^\pm(\ell^\infty_{2^N})$ to show that neither c_0 nor ℓ^∞ is UMD$^-$ or UMD$^+$. Clearly, this also implies that neither space is UMD.

Example 3.3.2. *The sequence spaces c_0 and ℓ^∞ are not UMD$^-$ or UMD$^+$ spaces.*

Proof. Fix $N \in \mathbb{N}$. By identifying sequences in $\ell^\infty_{2^N}$ with the first 2^N coordinates of sequences in c_0, $\ell^\infty_{2^N}$ is a closed subspace of c_0 (which is in turn a closed subspace of ℓ^∞). Using Proposition 3.2.3 and the previous lemma,

$$\beta_p^+(\ell^\infty) \geq \beta_p^+(c_0) \geq \beta_p^+(\ell^\infty_{2^N}) \geq c_p^+\sqrt{N}, \quad \beta_p^-(\ell^\infty) \geq \beta_p^-(c_0) \geq \beta_p^-(\ell^\infty_{2^N}) \geq c_p^-\sqrt{N}.$$

This holds for all $N \in \mathbb{N}$, so both pairs of randomized UMD constants are infinite and neither c_0 nor ℓ^∞ is UMD$^+$ or UMD$^-$. □

The space ℓ^1 leads to a slightly different situation. As we saw was the case for c_0 and ℓ^∞, we will find that ℓ^1 fails the UMD$^+$ property. However, unlike the other sequence spaces, ℓ^1 *is* a UMD$^-$ space.

Example 3.3.3. *The sequence space ℓ^1 is a UMD$^-$ space but is not a UMD$^+$ space.*

Proof. If ℓ^1 were UMD$^+$, then Propositions 3.2.2 and 3.2.5 would give that $\ell^\infty = (\ell^1)^*$ is UMD$^-$, but we showed that this is not the case in Example 3.3.2. Therefore, ℓ^1 is not a UMD$^+$ space.

For UMD$^-$, let $(f_n)_{n=1}^N$ be an $L^2(S;\ell^1)$-martingale for a σ-finite measure space $(S,\mathcal{A},\mu)$. Taking the $L^2(S;\ell^1)$-norm of f_N, we can then bound

$$\begin{aligned}
\left\|\sum_{n=1}^N df_n\right\|_{L^2(S;\ell^1)} &= \left\|\sum_{t=1}^\infty\left|\sum_{n=1}^N df_n(t)\right|\right\|_{L^2(S)} \\
&\le \sum_{t=1}^\infty\left\|\sum_{n=1}^N df_n(t)\right\|_{L^2(S)} \\
&= \sum_{t=1}^\infty\left\|\left(\sum_{n=1}^N |df_n(t)|^2\right)^{\frac{1}{2}}\right\|_{L^2(S)} \\
&\le \frac{1}{A_1}\sum_{t=1}^\infty\left\|\sum_{n=1}^N \varepsilon_n df_n(t)\right\|_{L^1(S\times\Omega)} \\
&= \frac{1}{A_1}\left\|\sum_{n=1}^N \varepsilon_n df_n\right\|_{L^1(S\times\Omega;\ell^1)} \\
&\le \frac{\kappa_{1,2}}{A_1}\left\|\sum_{n=1}^N \varepsilon_n df_n\right\|_{L^2(S\times\Omega;\ell^1)},
\end{aligned}$$

where we use the triangle inequality, Proposition 2.2.4, Khintchine's inequality, and the Kahane-Khintchine inequalities. It follows that ℓ^1 is a UMD$^-$ space. □

Note that this example also implies that any Banach space which is finitely representable in ℓ^1 is also a UMD$^-$ space, by the discussion in Subsection 3.2.4.

As ℓ^1 is not a UMD$^+$ space, it is not a UMD space. We conclude that UMD$^-$ is a strictly weaker property than UMD. It remains open whether UMD$^+$ is strictly weaker than UMD, as it is unknown whether there exist any UMD$^+$ spaces which are not also UMD$^-$ (hence UMD).

Intuitively, one can consider c_0 to be a barrier to both UMD$^-$ and UMD$^+$ and ℓ^1 to be a barrier to UMD$^+$ in the sense that we can embed them into many classical Banach spaces to demonstrate that they fail one or both of the randomized UMD properties.

For example, $C([0,1])$ and $L^\infty([0,1])$ contain isometric copies of c_0, so they are neither UMD$^-$ nor UMD$^+$. $L^1([0,1])$ contains an isometric copy of ℓ^1, so it is not a UMD$^+$ space. However, $L^1([0,1])$ is finitely representable in ℓ^1, so it is actually a UMD$^-$ space. We will generalize the example of $L^1([0,1])$ in Proposition 3.3.5.

3.3.2 L^p spaces

After proving Proposition 3.2.1, we mentioned that UMD, UMD$^+$, and UMD$^-$ spaces can be considered as particular generalizations of Hilbert spaces. Example 3.3.3 demonstrates that ℓ^1 is a UMD$^-$ spaces, so we know that UMD$^-$ spaces are an intermediate between Hilbert and general Banach spaces. It remains to show the existence of UMD$^+$ and UMD spaces which are not Hilbert spaces. The clearest examples are L^p spaces. First, we treat the UMD$^+$ property.

Proposition 3.3.4. *Let $(T, \mathcal{B}, \nu)$ be a σ-finite measure space and let X be a UMD$^+$ space. For all $p \in (1, \infty)$, $L^p(T; X)$ is a UMD$^+$ space with $\beta_p^+(L^p(T;X)) = \beta_p^+(X)$.*

Proof. Let $(df_n)_{n=1}^N$ be an $L^p(T;X)$-valued L^p-martingale difference sequence on a σ-finite measure space $(S, \mathcal{A}, \mu)$. Under the identification $L^p(S; L^p(T;X)) \cong L^p(T; L^p(S;X))$,

$$\mathbb{E}_{L^p(T;X)}[\cdot \mid \mathcal{F}_n] = \mathbb{E}_{L^p(T)}[\cdot \mid \mathcal{F}_n] \otimes I_X,$$

so for ν-almost all $t \in T$, $df_n(t)$ is an $L^p(S;X)$-martingale difference sequence. Therefore, we can use the UMD$^+$ property for X to write that

$$\left\| \sum_{n=1}^N \varepsilon_n df_n(t) \right\|_{L^p(S\times\Omega;X)} \leq \beta_p^+(X) \left\| \sum_{n=1}^N df_n(t) \right\|_{L^p(S;X)}$$

ν-almost everywhere.

Then, using Fubini's theorem in the first and final steps,

$$\begin{aligned}
\left\|\sum_{n=1}^{N}\varepsilon_n df_n\right\|_{L^p(S\times\Omega;L^p(T;X))} &= \left(\int_T \left\|\sum_{n=1}^{N}\varepsilon_n df_n(t)\right\|_{L^p(S\times\Omega;X)}^p \mathrm{d}\nu(t)\right)^{\frac{1}{p}} \\
&\leq \left(\int_T \beta_p^+(X)^p \left\|\sum_{n=1}^{N} df_n(t)\right\|_{L^p(S;X)}^p \mathrm{d}\nu(t)\right)^{\frac{1}{p}} \\
&= \beta_p^+(X)\left\|\sum_{n=1}^{N} df_n\right\|_{L^p(S;L^p(T;X))},
\end{aligned}$$

so $L^p(T;X)$ is a UMD$^+$ space with $\beta_p^+(L^p(T;X)) \leq \beta_p^+(X)$.

For equality, let $f \in L^p(T)$ with $\|f\|_{L^p(T)} = 1$. Then, $x \mapsto f \otimes x$ is an isometric embedding of X into $L^p(T;X)$, which implies that $\beta_p^+(X) \leq \beta_p^+(L^p(T;X))$. All together, we find that $L^p(T;X)$ is a UMD$^+$ space with $\beta_p^+(L^p(T;X)) = \beta_p^+(X)$. □

In particular, as Proposition 3.2.1 gives us that $\mathbb{R}$ and $\mathbb{C}$ are UMD spaces, the previous proposition implies that, for σ-finite T and $p \in (1,\infty)$, $L^p(T)$ is a UMD$^+$ space.

A similar statement (with similar proof) holds for the UMD$^-$ property. However, we actually have a broader result: for a UMD$^-$ space X, $L^1(T;X)$ is also a UMD$^-$ space. As we saw (in Example 3.3.3 and the following discussion), the same does not hold for the UMD$^+$ property, and hence for the standard UMD property.

Proposition 3.3.5. *Let $(T,\mathcal{B},\nu)$ be a σ-finite measure space and let X be a UMD$^-$ space. For all $p \in [1,\infty)$, $L^p(T;X)$ is a UMD$^-$ space with $\beta_p^-(L^p(T;X)) = \beta_p^-(X)$.*

Proof. First, suppose that $p \in (1,\infty)$; this case is very similar to the previous proposition. Let $(df_n)_{n=1}^N$ be an $L^p(T;X)$-valued L^p-martingale difference sequence on a σ-finite measure space $(S,\mathcal{A},\mu)$. Under the identification $L^p(S;L^p(T;X)) \cong L^p(T;L^p(S;X))$,

$$\mathbb{E}_{L^p(T;X)}[\cdot \mid \mathcal{F}_n] = \mathbb{E}_{L^p(T)}[\cdot \mid \mathcal{F}_n] \otimes I_X,$$

so for ν-almost all $t \in T$, $df_n(t)$ is an $L^p(S;X)$-martingale difference sequence. Therefore, we can use the UMD$^-$ property for X to write that

$$\left\|\sum_{n=1}^{N} df_n(t)\right\|_{L^p(S;X)} \leq \beta_p^-(X)\left\|\sum_{n=1}^{N}\varepsilon_n df_n(t)\right\|_{L^p(S\times\Omega;X)}$$

ν-almost everywhere.

Then, using Fubini's theorem in the first and final steps,

$$\begin{aligned}\left\|\sum_{n=1}^{N} df_n\right\|_{L^p(S;L^p(T;X))} &= \left(\int_T \left\|\sum_{n=1}^{N} df_n(t)\right\|_{L^p(S;X)}^p \mathrm{d}\nu(t)\right)^{\frac{1}{p}} \\ &\leq \left(\int_T \beta_p^-(X)^p \left\|\sum_{n=1}^{N} \varepsilon_n df_n(t)\right\|_{L^p(S\times\Omega;X)}^p \mathrm{d}\nu(t)\right)^{\frac{1}{p}} \\ &= \beta_p^-(X)\left\|\sum_{n=1}^{N} \varepsilon_n df_n\right\|_{L^p(S\times\Omega;L^p(T;X))}\end{aligned}$$

so $L^p(T;X)$ is a UMD$^-$ space with $\beta_p^-(L^p(T;X)) \leq \beta_p^-(X)$.

For equality, let $f \in L^p(T)$ with $\|f\|_{L^p(T)} = 1$. Then, $x \mapsto f \otimes x$ is an isometric embedding of X into $L^p(T;X)$, which implies that $\beta_p^-(X) \leq \beta_p^-(L^p(T;X))$. All together, we find that $L^p(T;X)$ is a UMD$^-$ space with $\beta_p^-(L^p(T;X)) = \beta_p^-(X)$.

For the case where $p = 1$, $L^1(T;X)$ is finitely representable in ℓ^1 (see [AK06, Proposition 11.1.7]), so using Example 3.3.3 and the discussion in Subsection 3.2.4, it is also a UMD$^-$ space. □

As with UMD$^+$, this implies that $L^p(T)$ is a UMD$^-$ space if T is σ-finite and $p \in [1,\infty)$. Combining the two propositions, the result holds when considering the standard UMD property as well.

Corollary 3.3.6. *Let $(T,\mathcal{B},\nu)$ be a σ-finite measure space and let X be a UMD space. For all $p \in (1,\infty)$, $L^p(T;X)$ is a UMD space with $\beta_p(L^p(T;X)) = \beta_p(X)$.*

In fact, the above propositions hold even for a non-σ-finite measure space $(T,\mathcal{B},\nu)$. In that case, we would use Lemma 2.2.5 to reduce to simple martingales which are adapted to filtrations of finite σ-algebras, after which the above argument would show that $L^p(T;X)$ is UMD$^+$ or UMD$^-$ for appropriate p.

3.4 R-boundedness

Now, we take a detour from our study of randomized UMD spaces to mention an important application of the theory that we have developed. As we have discussed, UMD and randomized UMD spaces are generalizations of Hilbert spaces that include a broader array of Banach spaces. Similarly, many results about families of bounded operators on Hilbert spaces extend to more general classes of Banach spaces if we replace uniform boundedness with a probabilistic analogue: R-boundedness.

This property of a family of operators is due to [BG94], which drew inspiration from [Bou86]. As we will see, R-boundedness resembles and is closely tied to the UMD property, but we frequently only need UMD$^+$ in order to show that a family of operators is R-bounded. Let us define the notion of an R-bounded family of operators.

Definition 3.4.1 (R-bounded)**.** *Let X and Y be Banach spaces and let $\mathcal{T}$ be a family of bounded linear operators from X to Y. $\mathcal{T}$ is said to be* R-bounded *if for some $p \in [1, \infty)$, there exists a finite constant $\mathcal{R} \geq 0$ such that*

$$\left\| \sum_{n=1}^{N} \varepsilon_n T_n x_n \right\|_{L^p(\Omega;Y)} \leq \mathcal{R} \left\| \sum_{n=1}^{N} \varepsilon_n x_n \right\|_{L^p(\Omega;X)}$$

for all finite sequences $(x_n)_{n=1}^N$ in X and $(T_n)_{n=1}^N$ in $\mathcal{T}$, where $(\varepsilon_n)_{n=1}^N$ is a Rademacher sequence on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

By the Kahane-Khintchine inequalities, the choice of $p \in [1, \infty)$ is arbitrary. For a particular choice of p, we denote by $\mathcal{R}_p(\mathcal{T})$ the infimum over all admissible $\mathcal{R}$.

In the remainder of this section, we show R-boundedness of the family of conditional expectation operators on a UMD$^+$ space induced by a filtration. This is a result of [Bou84, Bou86] extending the scalar case given by [Ste70, Theorem 8], but the proof below is due to [FW01, Lemma 34].

Proposition 3.4.2 (Vector-valued Stein inequality)**.** *Let X be a UMD$^+$ space and fix $p \in (1, \infty)$. Let $(S, \mathcal{A}, \mu)$ be a measure space with a σ-finite filtration $(\mathcal{F}_n)_{n=1}^N$ and denote by $\mathcal{E}$ the family of conditional expectation operators $\{\mathbb{E}[\cdot \mid \mathcal{F}_n]\}_{n=1}^N$. Then, $\mathcal{E}$ is R-bounded on $L^p(S;X)$ with constant $\mathcal{R}_p(\mathcal{E}) \leq \beta_p^+(X)$.*

That is, for all finite sequences $(f_n)_{n=1}^N$ in $L^p(S;X)$, we have

$$\left\| \sum_{n=1}^{N} \varepsilon_n \mathbb{E}[f_n \mid \mathcal{F}_n] \right\|_{L^p(S\times\Omega;X)} \leq \beta_p^+(X) \left\| \sum_{n=1}^{N} \varepsilon_n f_n \right\|_{L^p(S\times\Omega;X)},$$

where $(\varepsilon_n)_{n=1}^N$ is a Rademacher sequence on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. Consider the family of sub-σ-algebras $(\mathcal{G})_{n=1}^{2N}$ of $\mathcal{A} \otimes \mathcal{F}$ given by

$$\mathcal{G}_{2n-1} = \sigma(\mathcal{F}_n, \varepsilon_0, \ldots, \varepsilon_{n-1}), \quad \mathcal{G}_{2n} = \sigma(\mathcal{F}_n, \varepsilon_0, \ldots, \varepsilon_n)$$

for $n = 1, \ldots, N$. Define $F = \sum_{n=1}^N \varepsilon_n f_n$ and consider the martingale $(F_n)_{n=1}^{2N}$ given by $F_n = \mathbb{E}[F \mid \mathcal{G}_n]$. By independence,

$$F_{2m-1} = \sum_{n=1}^{m-1} \varepsilon_n \mathbb{E}[f_n \mid \mathcal{F}_m], \quad F_{2m} = \sum_{n=1}^{m} \varepsilon_n \mathbb{E}[f_n \mid \mathcal{F}_m],$$

so that

$$dF_{2n} = \varepsilon_n \mathbb{E}[f_n \mid \mathcal{F}_n].$$

Let $(\widetilde{\varepsilon}_n)_{n=1}^{2N}$ be another Rademacher sequence on another probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$. Then,

$$\begin{aligned}
\left\| \sum_{n=1}^N \varepsilon_n \mathbb{E}[f_n \mid \mathcal{F}_n] \right\|_{L^p(S\times\Omega;X)} &= \left\| \sum_{n=1}^N \widetilde{\varepsilon}_{2n} dF_{2n} \right\|_{L^p(S\times\Omega\times\widetilde{\Omega};X)} \\
&\le \left\| \sum_{n=1}^{2N} \widetilde{\varepsilon}_n dF_n \right\|_{L^p(S\times\Omega\times\widetilde{\Omega};X)} \\
&\le \beta_p^+(X) \| F_{2N} \|_{L^p(S\times\Omega;X)} \\
&\le \beta_p^+(X) \left\| \sum_{n=1}^N \varepsilon_n f_n \right\|_{L^p(S\times\Omega;X)},
\end{aligned}$$

where the inequalities correspond to Kahane's contraction principle, the UMD$^+$ property of X, and contractivity of the conditional expectation $\mathbb{E}\left[\cdot \mid \mathcal{G}_{2N}\right]$. □

The R-boundedness of families of conditional expectation operators has far-reaching applications even outside of the theory of martingales. For example, it implies that the heat semigroup on $L^p(\mathbb{R}^d; X)$ for a UMD$^+$ space X is also R-bounded with constant $\beta_p^+(X)$. See [HLN16, Corollary 21] for the details. Similarly, [HvP08] shows that the family of operators averaging over balls in $\mathbb{R}^d$ is R-bounded on $L^p(\mathbb{R}^d; X)$ with constant linear in $\beta_p^+(X)$ if X is a UMD$^+$ space. See [DHP03] and [KW04] for surveys of the theory as well as some applications to harmonic analysis and parabolic partial differential equations.

In this chapter, we defined the UMD, UMD$^+$, and UMD$^-$ properties, then discussed several constructions of such spaces as well as some canonical sequence and functions spaces which satisfy some, none, or all of the properties. We finished with a simple example in which the UMD$^+$ property is helpful for analysis. In the next section, we will study the geometric consequences of each of the three properties.

Chapter 4

Geometry of Banach Spaces

The UMD and randomized UMD (UMD^+ and UMD^-) properties are only three of many conditions that can be used to characterize the geometry of Banach spaces. In this chapter, we summarize some of the most commonly encountered geometric properties of Banach spaces and compare them to the randomized UMD properties. These properties have far-reaching implications for the geometry of Banach spaces even in settings without any clear connection to probability or martingales.

4.1 Reflexivity

For a Banach space X with bidual X^{**}, the canonical evaluation map $J : X \to X^{**}$ is given by $J(x)(x^*) = x^*(x)$ for all $x^* \in X^*$. We call X *reflexive* if this map is a homeomorphism. By the Hahn-Banach theorem, J is injective and preserves norms, so it suffices to prove that J is surjective. As we will show, this is the case if X is a UMD^+ space.

UMD spaces were first shown to be reflexive in [Mau75, Ald79]. Using similar methods, [Gar90] shows that the UMD^+ property is sufficient for a Banach space to be reflexive. Both arguments rely on the following alternative characterization of reflexivity of a Banach space due to [Jam64]. See [HvVW16, Theorem 4.3.2] for a recent proof of the theorem in this form and its application to show that the UMD property implies reflexivity, from which our proof is adapted.

Lemma 4.1.1 (James's theorem)**.** *Let X be a Banach space. The following are equivalent:*

(1) X is not reflexive.

(2) for some $0 < \theta < \frac{1}{2}$, there exist sequences $(x_n)_{n\in\mathbb{N}}$ in $\overline{B}_X$ and $(x_n^*)_{n\in\mathbb{N}}$ in $\overline{B}_{X^*}$ such that

$$x_i^*(x_j) = \begin{cases} \theta & \text{if } i \leq j \\ 0 & \text{if } i > j. \end{cases}$$

(3) for some $\theta > 0$, there exists a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X such that for all $N \in \mathbb{N}$,

$$d(\operatorname{conv}(x_n)_{n=1}^N, \operatorname{conv}(x_n)_{n=N+1}^\infty) \geq \theta.$$

Armed with James's theorem, we proceed to show that UMD$^+$ spaces are reflexive by constructing a martingale in any non-reflexive Banach space for which the UMD$^+$ property would fail. The construction of this particular martingale is originally from [Pis75], while the proof below using Pisier's martingale is due to [Gar90].

Proposition 4.1.2. *If X is a UMD$^+$ space, then X is reflexive.*

Proof. Suppose for the sake of contradiction that X is a UMD$^+$ space which is not reflexive. Fix $p \in (1, \infty)$ and $N \in \mathbb{N}$. By James's theorem, there exists $0 < \theta < \frac{1}{2}$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in $\overline{B}_X$ such that $d(A_n, B_n) \geq 2\theta$ for all $n \in \mathbb{N}$, where

$$A_n = \operatorname{conv}(\{x_1, \ldots, x_n\}), \quad B_n = \operatorname{conv}(\{x_{n+1}, x_{n+2}, \ldots\}).$$

Define the function $f_N : [0, 1) \to X$ by

$$f_N := \sum_{n=1}^{2^N} \mathbb{1}_{[(n-1)2^{-N}, n2^{-N})} x_n,$$

which is measurable with respect to $\mathcal{D}_N$, where $(\mathcal{D}_n)_{n=1}^N$ is the dyadic filtration generated by the intervals $I_n^N = [(n-1)2^{-N}, n2^{-N})$ for $n = 1, \ldots, 2^N$. Then, define the martingale $(f_n)_{n=1}^N$ by $f_n = \mathbb{E}[f_N \mid \mathcal{D}_n]$. On the interval I_j^n, f_n almost everywhere takes the value

$$y_j^n := \frac{1}{|I_j^n|} \int_{I_j^n} f_N(t)\,\mathrm{d}t = 2^{n-N} \sum_{k=(j-1)2^{N-n}+1}^{j2^{N-n}} x_k.$$

Therefore, $y_j^n \in \operatorname{conv}(\{x_{(j-1)2^{N-n}+1}, \ldots, x_{j2^{N-n}}\})$. As a consequence of James's theorem, for distinct $i, j = 1, \ldots, 2^n$, $\|y_j^n\|_X \leq 1$ (hence $\|f_n(t)\|_X \leq 1$ almost everywhere) and $\|y_i^n - y_j^n\|_X \geq 2\theta$.

As $I_j^{n-1} = I_{2j-1}^n \cup I_{2j}^n$, we can decompose $y_j^{n-1} = \frac{1}{2}(y_{2j-1}^n + y_{2j}^n)$. Therefore, for almost every $t \in I_{2j-1}^n$,

$$\|df_n(t)\|_X = \left\| y_{2j-1}^n - \frac{1}{2}(y_{2j-1}^n + y_{2j}^n) \right\|_X = \frac{1}{2}\left\| y_{2j-1}^n - y_{2j}^n \right\| \geq \theta$$

and for almost every $t \in I_{2j}^n$,

$$\|df_n(t)\|_X = \left\| y_{2j}^n - \frac{1}{2}(y_{2j-1}^n + y_{2j}^n) \right\| = \frac{1}{2}\left\| y_{2j}^n - y_{2j-1}^n \right\| \geq \theta.$$

Now, let $(\varepsilon_n)_{n=1}^N$ be a Rademacher sequence on a probability space Ω. We can compute

$$\begin{aligned}
\theta|a_n| &\leq \|a_n df_n\|_{L^p([0,1);X)} \\
&= \|a_n \varepsilon_n df_n\|_{L^p(\Omega\times[0,1);X)} \\
&\leq \left\| \sum_{n=1}^N a_n \varepsilon_n df_n \right\|_{L^p(\Omega\times[0,1);X)} \\
&\leq \|a\|_{\ell_N^\infty} \left\| \sum_{n=1}^N \varepsilon_n df_n \right\|_{L^p(\Omega\times[0,1);X)} \\
&\leq \beta_p^+(X)\|a\|_{\ell_N^\infty} \|f_N\|_{L^p([0,1);X)} \\
&\leq \beta_p^+(X)\|a\|_{\ell_N^\infty}
\end{aligned}$$

where we use $\|df_n(t)\|_X \geq \theta$ almost everywhere from above, Kahane's contraction principle, the UMD$^+$ property of X, and finally that $\|f_N(t)\|_X \leq 1$ almost everywhere also from above.

The computation above shows that the map $J : a \mapsto \sum_{n=1}^N a_n \varepsilon_n df_n$ is an isomorphic embedding of ℓ_N^∞ into $L^p(\Omega \times [0,1); X)$ with $\theta \leq \|J\| \leq \beta_p^+(X)$. By Proposition 3.2.2, it follows that $\beta_p^+(\ell_N^\infty) \leq \frac{1}{\theta}\beta_p^+(X)$. This holds for all $N \in \mathbb{N}$, and $\beta_p^+(\ell_N^\infty) \to \infty$ as $N \to \infty$ by Lemma 3.3.1, so $\beta_p^+(X)$ must be infinite and X must not be UMD$^+$. This is a contradiction, so we conclude that X must be reflexive. □

Recall that ℓ^1, ℓ^∞, and c_0 are not reflexive. The proposition above implies that they are not UMD$^+$ spaces, which confirms our conclusions in Examples 3.3.2 and 3.3.3. The same proposition does not hold if UMD$^+$ is replaced by UMD$^-$: ℓ^1 is a UMD$^-$ space, but it is not reflexive.

Actually, Proposition 4.1.2 can be strengthened using an existing result. In Subsection 3.2.4, we found that UMD$^+$ is a super-property in that it is preserved under finite representability. That is, if a Banach space Y is finitely representable in a UMD$^+$ space X, then it is also UMD$^+$. Using Proposition 4.1.2, it follows that Y is reflexive. All together, we find that any Banach space Y which is finitely representable in a UMD$^+$ space X is reflexive, a property of X which we call *super-reflexivity*. This gives the following corollary.

Corollary 4.1.3. *If X is a UMD$^+$ space, then X is super-reflexive.*

Now that we have shown that the UMD$^+$ property implies super-reflexivity, it is natural to ask whether the converse holds (i.e. if UMD$^+$ and super-reflexivity are equivalent properties). The answer is no. In [Bou83], Bourgain provides an example of a super-reflexive Banach lattice which is not UMD$^+$. Garling adapts this example in [Gar90, Theorem 4] to construct a super-reflexive Banach lattice which is not UMD$^-$ either. In [Qiu12], Qiu provides a considerably simpler example of a non-UMD$^+$ super-reflexive Banach lattice.

4.2 K-convexity

The next property that we consider is K-convexity. This notion was first introduced by [MP76] in order to obtain duality results for cotype similar to those of type (see the succeeding section for the details). The following definition characterizes K-convex spaces as those for which Rademacher projections are uniformly bounded. As we will see, K-convexity follows from the UMD$^+$ property.

Definition 4.2.1 (K-convex)**.** *A Banach space X is called K-convex if, for some $p \in (1, \infty)$,*

$$K_p(X) := \sup_{N \geq 1} \|\mathrm{Rad}_N\|_{L^p(\Omega;X) \to L^p(\Omega;X)}$$

is finite, where

$$\mathrm{Rad}_N(f) := \sum_{n=1}^{N} r_n \mathbb{E}\left[r_n f\right]$$

for a real Rademacher sequence $(r_n)_{n=1}^N$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

By the Kahane-Khintchine inequalities, the choice of p is arbitrary. That is, $K_p(X)$ is finite if and only if $K_q(X)$ is finite for any $p, q \in (1, \infty)$. Of course, different choices of p lead to different quantitative characterizations $K_p(X)$ of K-convex spaces. Note that the definition does not make use of any complex structure on X, so $K_p(X) =$

$K_p(X_\mathbb{R})$ for a complex Banach space X. We could have replaced $(r_n)_{n=1}^N$ with a complex Rademacher sequence to obtain the same property with different constants $K_p(X)$ (in which case $K_p(X) \neq K_p(X_\mathbb{R})$).

We begin with a reduction of the functions that need to be considered as inputs to Rad_N from all of $L^p(\Omega; X)$ to a subset which is measurable with respect to a certain sub-σ-algebra. This reduction will make it easier to show that K-convexity follows from the UMD$^+$ property because only the Rademacher projections of this smaller class of functions need be considered.

Lemma 4.2.2. *Let X be a Banach space and fix $p \in (1, \infty)$. Let $(r_n)_{n=1}^N$ be a real Rademacher sequence on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider the sub-σ-algebra given by $\mathcal{F}_N = \sigma(r_1, \ldots, r_N)$. Then,*

$$\left\| \mathrm{Rad}_N|_{L^p(\Omega, \mathcal{F}_N; X)} \right\| = \|\mathrm{Rad}_N\|.$$

Proof. For any $f \in L^p(\Omega; X)$ and all $n = 1, \ldots N$,

$$\mathbb{E}\left[r_n f\right] = \mathbb{E}\left[\mathbb{E}\left[r_n f \mid \mathcal{F}_N\right]\right] = \mathbb{E}\left[r_n \mathbb{E}\left[f \mid \mathcal{F}_N\right]\right].$$

Therefore, $\mathrm{Rad}_N(f) = \mathrm{Rad}_N(\mathbb{E}\left[f \mid \mathcal{F}_N\right])$, so it suffices to consider only functions which are measurable with respect to $\mathcal{F}_N$. □

Now, for fixed $N \in \mathbb{N}$, consider the probability space $D = \{-1, 1\}^N$ equipped with the uniform probability measure μ. The coordinate mappings $r_n(\omega) = \omega_n$ form a real Rademacher sequence on (D, μ), using which we define the Walsh system $(w_\alpha)_\alpha$ by

$$w_\alpha = \prod_{n \in \alpha} r_n$$

for any subset $\alpha \subseteq \{1, \ldots, N\}$. The following lemma allows us to represent every $f : D \to X$ using the Walsh system.

Lemma 4.2.3. *Let X be a Banach space and fix $p \in (1, \infty)$. For every function $f : D \to X$, we can write*

$$f = \sum_{\alpha \subseteq \{1, \ldots, N\}} w_\alpha \mathbb{E}\left[w_\alpha f\right]. \tag{4.2.1}$$

Proof. It is immediate that the Walsh system $(w_\alpha)_\alpha$ is orthonormal in the Hilbert space $L^2(D)$. As $L^2(D)$ has dimension 2^N and $(w_\alpha)_\alpha$ has 2^N elements, it follows that $(w_\alpha)_\alpha$ is an orthonormal basis for $L^2(D)$. The claim follows. □

Proposition 4.2.4. *If X is a UMD$^+$ space, then X is K-convex with $K_p(X) \leq C_{\mathbb{K}}\beta_p^+(X)$ for all $p \in (1,\infty)$, where $C_{\mathbb{R}} = 1$ and $C_{\mathbb{C}} = \frac{\pi}{2}$.*

Proof. First, suppose that X is a real Banach space and fix $N \in \mathbb{N}$. Using the lemmas, it suffices to show that for any $f \in L^p(D;X)$ of the form given in eq. (4.2.1),

$$\|\mathrm{Rad}_N(f)\|_{L^p(D;X)} \leq \beta_p^+(X)\|f\|_{L^p(D;X)},$$

from which the claim follows.

For $n = 1,\dots,N$, define

$$A_n := \{\alpha \subseteq \{1,\dots,N\} \mid \max\alpha = n\}$$

and

$$d_n := \sum_{\alpha\in A_n} w_\alpha \mathbb{E}\left[w_\alpha f\right],$$

so that

$$f = \sum_{n=1}^{N} d_n.$$

As $(r_n)_{n=1}^N$ is an independent Rademacher sequence, $\mathbb{E}\left[d_n \mid \sigma(r_1,\dots,r_{n-1})\right] = 0$. It follows that $(d_n)_{n=1}^N$ is a martingale difference sequence. Also, when $m \neq n$, $\mathbb{E}\left[r_n w_\alpha\right] = 0$ for all $\alpha \in A_m$, which implies that $\mathbb{E}\left[r_n f\right] = \mathbb{E}\left[r_n d_n\right]$.

Let $(\widetilde{r}_n)_{n=1}^N$ be another real Rademacher sequence on another probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$.

Then, for all $f \in L^p(D;X)$,

$$\begin{aligned}
\mathbb{E}\|\mathrm{Rad}_N(f)\|^p &= \mathbb{E}\left\|\sum_{n=1}^N r_n \mathbb{E}\left[r_n f\right]\right\|^p \\
&= \widetilde{\mathbb{E}}\left\|\sum_{n=1}^N \widetilde{r}_n \mathbb{E}\left[r_n d_n\right]\right\|^p \\
&= \widetilde{\mathbb{E}}\left\|\mathbb{E}\sum_{n=1}^N \widetilde{r}_n r_n d_n\right\|^p \\
&\leq \widetilde{\mathbb{E}}\mathbb{E}\left\|\sum_{n=1}^N \widetilde{r}_n r_n d_n\right\|^p \\
&= \mathbb{E}\widetilde{\mathbb{E}}\left\|\sum_{n=1}^N \widetilde{r}_n r_n d_n\right\|^p \\
&= \mathbb{E}\widetilde{\mathbb{E}}\left\|\sum_{n=1}^N \widetilde{r}_n d_n\right\|^p \\
&\leq (\beta_p^+(X))^p \mathbb{E}\left\|\sum_{n=1}^N d_n\right\|^p \\
&= (\beta_p^+(X))^p \mathbb{E}\|f\|^p,
\end{aligned}$$

where we use Fubini's theorem, that $(r_n(\omega)\widetilde{r}_n)_{n=1}^N$ is also a real Rademacher sequence on $\widetilde{\Omega}$, and the UMD$^+$ property. We conclude that

$$\|\mathrm{Rad}_N\|_{L^p(D;X)\to L^p(D;X)} \leq \beta_p^+(X),$$

so X is K-convex and $K_p(X) \leq \beta_p^+(X)$ as long as X is a real Banach space.

If X is a complex Banach space, then by Proposition 3.2.4,

$$K_p(X) = K_p(X_{\mathbb{R}}) \leq \beta_p^+(X_{\mathbb{R}}) \leq \frac{\pi}{2}\beta_p^+(X),$$

so X is K-convex and $K_p(X) \leq \frac{\pi}{2}\beta_p^+(X)$ as desired. $\square$

Unlike the UMD$^+$ property, the UMD$^-$ property does not necessarily imply K-convexity. Example 3.3.3 gives that ℓ^1 is a UMD$^-$ space, but it is not K-convex (see [HvVW17, Corollary 7.1.10]).

4.3 Type and cotype

Next, we study type and cotype, which quantitatively characterize the geometry of Banach spaces using the behavior of Rademacher sums. The history of type and cotype is described in [Mau03], while [PW98, AK06, HvVW17] are reference texts with proofs of many of the theorems that we mention below. As with K-convexity, we will see that type and cotype are linked (qualitatively, at least) with the randomized UMD properties.

Definition 4.3.1 (Type and cotype). *Let X be a Banach space, $p \in [1,2]$ and $q \in [2,\infty]$. The space X is said to have* type p *if there exists a finite constant $\tau \geq 0$ such that*

$$\left\| \sum_{n=1}^{N} \varepsilon_n x_n \right\|_{L^p(\Omega;X)} \leq \tau \left(\sum_{n=1}^{N} \|x_n\|_X^p \right)^{\frac{1}{p}}$$

for all finite sequences $(x_n)_{n=1}^N$ in X, where $(\varepsilon_n)_{n=1}^N$ is a Rademacher sequence on a probability space Ω.

The space X is said to have cotype q *if there exists a finite constant $c \geq 0$ such that*

$$\left(\sum_{n=1}^{N} \|x_n\|_X^q \right)^{\frac{1}{q}} \leq c \left\| \sum_{n=1}^{N} \varepsilon_n x_n \right\|_{L^q(\Omega;X)}$$

under the same conditions.

We say that a Banach space has non-trivial type if it has type p for some $p > 1$ and finite cotype if it has type q for some $q < \infty$. Note that type p implies type $\widetilde{p}$ for all $\widetilde{p} \in [1,p]$ and cotype q implies cotype $\widetilde{q}$ for all $\widetilde{q} \in [q,\infty]$. The König-Tzafriri theorem in [KT81] stipulates that non-trivial type implies finite cotype. However, the converse does not hold: ℓ^1 has cotype 2, but no non-trivial type (see [HvVW17, Corollary 7.1.10]).

We denote by $\tau_p(X)$ and $c_q(X)$ the infima over all admissible τ and c, respectively. By the Kahane-Khintchine inequalities, the exponents of the Rademacher sums can be replaced by r and $1/r$ for arbitrary $r \in [1,\infty)$, except for the case $q = \infty$ (although this does lead to different constants).

Kwapień's theorem in [Kwa72] specifies that a Banach space is isomorphic to a Hilbert space if and only if it has type 2 and cotype 2. For that reason, type and cotype can be thought of as quantitative characterizations of how close or far a Banach space is from being a Hilbert space.

We proved in Proposition 3.2.1 that all Hilbert spaces are UMD spaces, so randomized UMD spaces can be considered as certain generalizations of Hilbert spaces. Therefore, we might hope for the UMD$^+$ and UMD$^-$ properties to imply at least non-trivial type and finite cotype. Indeed, this holds for all but UMD$^-$ and non-trivial type.

Corollary 4.3.2. *If X is a UMD$^+$ space, then X has type p for some $p > 1$.*

This is a simple corollary of Proposition 4.2.4 and the deep fact that K-convexity is equivalent to non-trivial type, a result of [Pis82]. As with K-convexity, the same is not true for UMD$^-$ because ℓ^1 is a UMD$^-$ space per Example 3.3.3, but has no non-trivial type.

Corollary 4.3.3. *If X is a UMD$^+$ space or a UMD$^-$ space, then X has cotype q for some $q < \infty$.*

Proof. For the first claim, if X is a UMD$^+$ space, the previous corollary implies that X has non-trivial type. Non-trivial type implies finite cotype (the König-Tzafriri theorem of [KT81]), so it follows that X has finite cotype as well.

For the second claim, if X is a UMD$^-$ space, ℓ^∞ is not finitely representable in it (otherwise ℓ^∞ would also be UMD$^-$, but Example 3.3.2 gives that it is not). This is equivalent to X having finite cotype by [MP76] (see [AK06, Theorem 11.1.4] for an English proof of this fact). □

There is also an interesting duality comparison between UMD$^+$/UMD$^-$ and type/cotype. In Proposition 3.2.5, we proved that the dual and predual of a UMD$^+$ space are UMD$^-$. However, we also saw that this is not always reversible: ℓ^1 is UMD$^-$, but neither its dual ℓ^∞ nor its predual c_0 are UMD$^+$.

In much the same way, if X has type p, then X^* has cotype p', but this is also not always reversible. The same example can be used: ℓ^1 has cotype 2, but neither its dual ℓ^∞ nor its predual c_0 have non-trivial type. See [HvVW17, Corollary 7.1.10, Proposition 7.1.13] for proofs of these facts.

There is still a missing piece to this puzzle. As mentioned earlier, [KT81] shows that non-trivial type implies finite cotype, but the converse does not always hold (ℓ^1 has cotype 2 but no non-trivial type). We know that UMD$^-$ does not always imply UMD$^+$, but there is no counterpart to the König-Tzafriri theorem: it is an open question whether UMD$^+$ implies UMD$^-$ (and therefore UMD).

4.4 Martingale type and cotype

Martingale type and cotype are variants of type and cotype which replace Rademacher sequences with martingale difference sequences. First introduced in [Pis75], they have many connections with convexity and smoothness properties of Banach spaces. Most notably, a Banach space X has martingale type p if and only if it admits an equivalent p-smooth norm and martingale cotype q if and only if it admits an equivalent q-convex norm.

The definitions below show quite clearly that martingale type and cotype imply type and cotype. We will see that the converse holds for UMD$^-$ and UMD$^+$ spaces, respectively.

Definition 4.4.1 (Martingale type and cotype)**.** *Let X be a Banach space, $p \in [1,2]$ and $q \in [2,\infty]$. The space X is said to have* martingale type p *if there exists a finite constant $\tau \geq 0$ such that*

$$\left\| \sum_{n=1}^{N} df_n \right\|_{L^p(S;X)} \leq \tau \left(\sum_{n=1}^{N} \|df_n\|_{L^p(S;X)}^p \right)^{\frac{1}{p}}$$

for any $L^p(S;X)$-martingale difference sequence $(df_n)_{n=1}^N$ on a σ-finite measure space $(S,\mathcal{A},\mu)$.

The space X is said to have martingale cotype q *if there exists a finite constant $c \geq 0$ such that*

$$\left(\sum_{n=1}^{N} \|df_n\|_{L^q(S;X)}^q \right)^{\frac{1}{q}} \leq c \left\| \sum_{n=1}^{N} df_n \right\|_{L^q(S;X)}$$

for any $L^q(S;X)$-martingale difference sequence $(df_n)_{n=1}^N$ on a σ-finite measure space $(S,\mathcal{A},\mu)$, with the left-hand side replaced with a supremum in the case $q=\infty$.

We denote by $\tau_p^{\text{mart}}(X)$ and $c_q^{\text{mart}}(X)$ the infima over all admissible τ and c, respectively.

By choosing the martingale difference sequence $df_n = \varepsilon_n x_n$ for a Rademacher sequence $(\varepsilon_n)_{n=1}^N$ and finite sequence $(x_n)_{n=1}^N$ in X, we find that martingale type p (resp. martingale cotype q) implies type p (resp. cotype q), with

$$\tau_p(X) \leq \tau_p^{\text{mart}}(X), \quad c_q(X) \leq c_q^{\text{mart}}(X).$$

According to the following proposition, the converses of these implications hold for UMD$^-$ and UMD$^+$ spaces, respectively.

Proposition 4.4.2. *Let $p \in (1,2]$ and $q \in [2,\infty)$.*

(1) If a UMD$^-$ space X has type p, then X also has martingale type p with $\tau_p^{\mathrm{mart}}(X) \leq \beta_p^-(X)\tau_p(X)$.

(2) If a UMD$^+$ space X has cotype q, then X also has martingale cotype q with $c_q^{\mathrm{mart}}(X) \leq \beta_q^+(X)c_q(X)$.

Proof. For the first claim, using that X is a UMD$^-$ space, then that it has type p, we obtain

$$\begin{aligned}\left\|\sum_{n=1}^N df_n\right\|_{L^p(S;X)} &\leq \beta_p^-(X)\left\|\sum_{n=1}^N \varepsilon_n df_n\right\|_{L^p(S\times\Omega;X)} \\ &\leq \beta_p^-(X)\tau_p(X)\left(\sum_{n=1}^N \|df_n\|_{L^p(S;X)}^p\right)^{\frac{1}{p}},\end{aligned}$$

so X has martingale type p with $\tau_p^{\mathrm{mart}}(X) \leq \beta_p^-(X)\tau_p(X)$.

For the second claim, using that X has cotype q, then that it is a UMD$^+$ space, we obtain

$$\begin{aligned}\left(\sum_{n=1}^N \|df_n\|_{L^q(S;X)}^q\right)^{\frac{1}{q}} &\leq c_q(X)\left\|\sum_{n=1}^N \varepsilon_n df_n\right\|_{L^q(S\times\Omega;X)} \\ &\leq c_q(X)\beta_q^+(X)\left\|\sum_{n=1}^N df_n\right\|_{L^q(S;X)},\end{aligned}$$

so X has martingale cotype q with $c_q^{\mathrm{mart}}(X) \leq \beta_q^+(X)c_q(X)$. □

Although martingale type and cotype can be strictly stronger properties than type and cotype in the general Banach space setting, we have shown that they are equivalent to their non-martingale counterparts for UMD$^-$ and UMD$^+$ spaces. By the geometric characterizations of martingale type and cotype from the beginning of this section, this implies that a UMD$^-$ space X has type p if and only if it admits an equivalent p-smooth norm and a UMD$^+$ space X has cotype q if and only if it admits an equivalent q-convex norm.

In this chapter, we demonstrated the connections tying the UMD$^+$ and UMD$^-$ properties to reflexivity, K-convexity, type/cotype, and martingale type/cotype. In doing so, we have enabled the use of these geometric tools when performing analysis on UMD^+ or UMD^- spaces. Far from being purely probabilistic characterizations of Banach spaces, UMD^+ and UMD^- carry with them deep ideas about the geometry of Banach spaces.

Chapter 5

Conclusion

We began our study of UMD and randomized UMD spaces with simple definitions characterizing the behavior of martingales and Rademacher difference sequences. Using the theory of martingales in Banach spaces which we built in Chapter 2 and armed with several constructions and counterexamples of UMD and randomized UMD spaces from Chapter 3, we were able to show in Chapter 4 that the purely probabilistic definitions of the UMD and randomized UMD properties have sweeping consequences for the geometry and analytical properties of Banach spaces.

This connection linking the behavior of probabilistic objects such as Rademacher sequences and martingales with the geometry of Banach spaces has been the central theme of this paper. While we have mentioned several other sources which discuss the analytical significance of the UMD property, none distinguish the UMD^+ and UMD^- properties from their non-randomized counterpart. Our main contribution has been the treatment of the randomized UMD properties independently of the UMD property, especially with regards to their geometric consequences.

We finish our study with a brief discussion of three major open questions relating to the randomized UMD properties.

The most important open question in the theory of the randomized UMD spaces is whether the UMD^+ property implies the UMD property. As we showed with Example 3.3.3, there exist UMD^- spaces which do not satisfy the UMD^+ property, so UMD is strictly stronger than UMD^-. However, a similar result has not been shown for UMD and UMD^+. There are currently no known UMD^+ spaces which are not also UMD^- (and thus UMD). This motivates the following question, first posed in

[Gei99].

Open Problem 1. *Does the UMD$^+$ property imply the UMD property?*

[Gei99, Corollary 5] presents some evidence which indicates that the UMD property may be strictly stronger than UMD$^+$: there is no general linear bound relating $\beta_p(X)$ to $\beta_p^+(X)$. One strategy for answering this question might be to compare the UMD$^+$ property with the boundedness of the Hilbert transform, which is equivalent with the UMD property.

Another major open problem in the theory is whether showing that a Banach space is UMD$^+$ or UMD$^-$ requires checking only that the defining inequalities hold for Walsh-Paley martingales. As we briefly mentioned in Section 3.1, one way to show that the UMD property is consistent regardless of one's choice of $p \in (1, \infty)$ is to reduce to Walsh-Paley martingales, for which the independence from choice of p is much easier to see. It is unknown whether the same reduction can be made for UMD$^+$ or UMD$^-$.

Open Problem 2. *Are the UMD$^+$ and UMD$^-$ properties implied by the corresponding notions when one restricts the defining inequalities to Walsh-Paley martingales?*

For the UMD$^-$ property, it is known that restricting the defining inequalities to Walsh-Paley martingales does lead to a different constant (which is denoted by $\beta_p^{-,\Delta}(X)$). As described in [CV11], this holds even for $X = \mathbb{R}$: it follows from [Bur91, Theorem 3.3] and [Hit94, Theorem 1.1] that $\beta_p^{-,\Delta}(\mathbb{R}) \neq \beta_p^-(\mathbb{R})$ for certain $p \in (1, \infty)$.

We also mentioned in Section 3.1 the result of [Gar86] that

$$\hbar_p(X) \leq \beta_p^+(X)\beta_p^-(X) \leq \beta_p(X)^2,$$

where $\hbar_p(X) \equiv \|H\|_{L^p(\mathbb{R};X) \to L^p(\mathbb{R};X)}$ is the norm of the Hilbert transform on $L^p(\mathbb{R}; X)$. It has long been conjectured that there is actually an estimate for the norm of the Hilbert transform of the form

$$\hbar_p(X) \leq c\beta_p(X)$$

for some constant $c > 0$. This is the content of the following open problem.

Open Problem 3. *Does there exist a linear bound of the form $\hbar_p(X) \leq c\beta_p(X)$ or even $\hbar_p(X) \leq c\beta_p^+(X)$?*

If the second inequality is true, then it would follow that the UMD and UMD$^+$ properties are equivalent, using that the UMD property is equivalent to boundedness of the Hilbert transform. This problem has been discussed in [Bur01, Section 3] and [PW98, 8.8.2] with some partial results, but remains open.

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